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FULLY CLOSED MAPS, SCANNABLE SPECTRA AND CARDINALITY OF HEREDITARILY SEPARABLE SPACES

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In this paper we study the notions of scannable spectrum and roll of a spectral tree, which appeared in slightly different form in [3] and [5] respectively. One of the main results: scannable spectra necessarily have fully closed projections and spectra of length $\leqslant\omega$ with fully closed projections are scannable. The technique of scannable spectra is used, when we study the new class of fully separable spaces. We prove that the statement – every fully separable almost perfectly normal compact space has cardinality $\leqslant\mathfrak{c}$ – is independent of ZFC.

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fully closed map	fully separable space
roll of a spectral tree	almost perfectly normal space
scannable spectrum	independent of *ZFC*

1. Introduction

All spaces, considered in this paper, are compact Hausdorff. All given maps are onto.

A simple map $f: X \to Y$, that is a map, which has at most one non-trivial inverse image $f^{-1}y$, is the simplest example of a fully closed map. A fibre-product $f: X \to Y$ of a family $\{X_\alpha \xrightarrow{f_\alpha} Y\}$ consisting of simple maps, gives us the most general example of a fully closed map (Corollary 4.16).

Every map $Y \xleftarrow{f} X$ can be considered as an inverse spectrum of length 2. A family $T = \{Y \xleftarrow{f_\alpha} X_\alpha\}$ is a simplest example of a spectral tree and its fibre-product $Y \xleftarrow{f} X$ is a roll of the spectral tree T.

An inverse spectrum S is scannable if it is a roll $S(T)$ of a simple spectral tree T. A scanning of S is a saturated spectral tree $T = T(S)$, such that $S = S(T)$.

In this paper we give a detailed description of fully closed maps and scannable spectra. A scannable spectrum is virtually the same as "разворачиваемый спектр", which was introduced in [3]. Then the word "разворачиваемый", was unsatisfactorily translated as upturnable [5]. A roll is virtually the same as "свертка", which was introduced in [5] and [6]. This was translated as a convolution [5].

One of the main results of this paper states: Scannable spectra necessarily have fully closed projections and spectra of length $\leq\omega$ with fully closed projections are scannable.

The notions of a scannable spectrum and a roll of a spectral tree were used in [3, 4, 5, 6], where various examples of topological spaces were constructed.

1.1. Remark. The axiom Φ, which was assumed in [4] and [6], is equivalent to Jensen's principle $\diamondsuit$ [7]. Indeed, clearly $\diamondsuit$ implies Φ and Φ implies both CH and Ostaszewski's principle $\clubsuit$ [9]. On the other hand, K. Devlin observed that CH+ $\clubsuit \Rightarrow \diamondsuit$ (see [14]).

1.2. Remark. The principle Δ_κ and the axiom F from [5] are false as K. Kunen justly remarked. They lack one condition, which disappeared, when the paper was abridged in the absence of the author. A correct formulation is the following.

Let κ be a regular cardinal and $T\subset\mathcal{P}(\kappa)$. The set T is called normal if there is an increasing sequence $\{T_\alpha\colon \alpha<\kappa\}$ such that

(a) $T_\alpha\subset T$ and $|T_\alpha|=|\alpha|$ for every $\alpha<\kappa$.

(b) $T_\alpha=\bigcup\{T_{\alpha'}\colon \alpha'<\alpha\}$ for every limit ordinal $\alpha<\kappa$.

(c) $T|\alpha\equiv\{t\cap\alpha : t\in T\}=T_\alpha|\alpha$ for every $\alpha<\kappa$.

Principle $\Delta_\kappa(B)$. For every regular cardinal κ and every stationary set $B\subset\kappa$ there is a sequence $\langle\tau_\alpha : \alpha\in B\rangle$, $\tau_\alpha\subset\mathcal{P}(\alpha)$ such that for every normal set $T\subset\mathcal{P}(\kappa)$ the set $\{\alpha : \tau_\alpha=T|\alpha\}$ is stationary in κ.

Axiom F is equivalent to $\Delta_{\omega_1}(\omega_1)$.

It is not difficult, using a bijection $\omega_1\to\omega_1\times\omega_1$, to prove that axiom F is equivalent to Jensen's principle $\diamondsuit$. The situation with $\Delta_\kappa(B)$ is analogous. Thus the principle $\diamondsuit$ was in fact used in [5] for a construction of hereditarily separable hereditarily normal compact space X with cardinality $2^{\mathfrak{c}}$.

The space X, just mentioned, has the following properties:

(1) X is fully separable (see Definition 6.7);

(2) every closed perfect subset of X is a G_δ-set;

(3) if $x\in\overline{F_i\backslash\{x\}}$, $i=1,2$, where F_i are closed, then there is a closed G_δ-set F such that $x\in F\subset F_1\cap F_2$.

Spaces with properties (2) and (3) will be called almost perfectly normal. This terminology is justified by the following:

1.3. Proposition. *Every compact first countable almost perfectly normal space is perfectly normal.*

Proof. If F is closed and uncountable, then its set of condensation points, denoted F^*, is closed and (by first countability) perfect. By property (2) we may write $F\backslash F^*$ as $\bigcup_{n=1}^\infty F_n$ with each F_n closed. By definition of F^* each F_n is countable and so the Proposition follows from the next Lemma which will be of interest to us later.

1.4. Lemma. *If A is a G_δ and B is a countable set of points of countable character, then $A \cup B$, if compact, is a G_δ.*

Proof. Let $A = \bigcap G_n$ for some descending sequence of open sets; let $B_n = B \setminus G_n$. Let $\{B_{n,m}\}_{m=1}^{\infty}$ be a neighbourhood base of the compact set B_n. Choose open sets G_m^* inductively so that $B_m \subset G_m^*$ and for $n < m$

$$G_m^* \setminus G_n \subset B_{n,m}.$$

This is possible since every $b \in B_m$ has a neighbourhood $G(b)$ satisfying

$$G(b) \subset G_n \qquad \text{if } b \in G_n \ (n < m),$$

$$G(b) \subset B_{n,m} \qquad \text{if } b \in B_{n,m} \ (n < m).$$

Now $A \cup B = \bigcap (G_n \cup G_n^*)$, for if $x \notin A \cup B$ then for some N, $x \notin G_N$. Moreover, $x \notin B_{N,k}$ for some $k \geqslant N$. But $G_k^* \setminus G_k \subset G_N \cup (G_k^* \setminus G_N) \subset G_N \cup B_{N,k}$, hence $x \notin G_k^* \cup G_k$.

In the last part of the paper we prove:

Theorem 6.13 (MA + $\neg$CH). *Every fully separable almost perfectly normal space is perfectly normal and so has cardinality* $\leqslant \mathbf{c}$.

Thus the statement:

Every fully separable almost perfectly normal space has cardinality $\leqslant \mathbf{c}$ is independent of ZFC.

The technique of scannable spectra is used essentially. We introduce the class of C-spaces, which contains all fully separable spaces, and in fact prove Theorem 6.13 for almost perfectly normal separable C-space X with $\mathrm{cc}(X) \leqslant \omega$.

The class of all C-spaces is much wider than the class of all fully separable spaces. For example, there are many C-spaces, which do not satisfy the Souslin condition. If the Souslin continuum exists, then there exists a non-separable perfectly normal C-space.

The following questions arise:

Problem 1. It is true that every C-space X with $\mathrm{cc}(X) \leqslant \omega$ is fully separable?

Problem 2. It is true that every fully separable (almost perfectly normal) space has cardinality $\leqslant 2^{\omega_1}$?

Problem 3. Is is true that every hereditarily separable (hereditarily normal) space has cardinality $\leqslant 2^{\omega_1}$?

A positive solution of Problem 3 would give us that the statement:

Every hereditarily separable hereditarily normal compact space has cardinality $\leqslant \mathbf{c}$.

is independent of ZFC (see [5]). At present we can state this only if we replace "$\leq \mathfrak{c}$" by "$<2^{\mathfrak{c}}$" (see [5] and [11]).

2. Fully closed maps

2.1. Definition [1]. A map $f: X \to Y$ is called *fully closed at a point* $y \in Y$ if for any finite covering $\{U_i : i = 1, \dots, s\}$ of its inverse image $f^{-1}y$ by sets open in X, the set $\{y\} \cup (\bigcup_{i=1}^{s} f^{\#} U_i)$ is open in Y. [Recall that if $A \subset X$, then $f^{\#}A = \{fa : f^{-1}fa \subset A\}$.] We shall also say, that $f: X \to Y$ is fully closed at a point $x \in X$, if f is fully closed at fx. If $f: X \to Y$ is fully closed at every point $y \in Y$, then f is called fully closed. We note that $f^{\#}U$ is open for U open when f is fully closed.

Example. To fix the reader's ideas, it may be helpful to refer to the example with X the lexicographic square, Y the interval $[0, 1]$ and f the projection $(x, y) \to x$. It is easily verified that f is fully closed. The relevance of this observation is that if X is thought of as a "fattened up" version of Y (each point of Y being replaced by a segment) then the "collapsing map" f is particularly well-behaved. Indeed, we shall soon see that, since f is fully closed, X may be obtained from a family of simple fattenings of Y in which only one point of Y is replaced by a segment and the other points are left alone. We have already noted in the introduction that simple maps are fully closed. Our general objective is to build spaces by an inductive process (formalized in the notion of spectral trees) whereby spaces are fattened up step by step so that the relevant "collapsings" form a transitive system of fully closed maps.

2.2. Lemma [2]. *Let $f: X \to Y$ and $g: Y \to Z$ be maps, such that gf is fully closed. Then g is also fully closed.*

2.3. Lemma. *A map $f: X \to Y$ is fully closed at a point $y \in Y$ if and only if for every open $U \subset X$ the set $U^y \equiv (U \cap f^{-1}y) \cup f^{-1}f^{\#}U$ is also open.*

Necessity is proved in [1]. Sufficiency is proved in [12].

2.4. Lemma [6]. *Let $f: X \to Y$ and $g: Y \to Z$ be maps, such that gf is fully closed at the point $z \in Z$ and let $y \in g^{-1}z$. If either $|g^{-1}gy| = 1$ or $|f^{-1}y| = 1$, then f is fully closed at y.*

Let $f: X \to Y$ be a fully closed map and let

$$\Phi = \{f^{-1}y \xrightarrow{\phi_y} Z_y : y \in Y\}$$

be a family of fully closed maps. Let

$$\mathcal{M}^{\phi_y} = \{\phi_y^{-1}z : z \in Z_y\} \quad \text{and} \quad \mathcal{M}^{\Phi} = \bigcup\{\mathcal{M}^{\phi_y} : \phi_y \in \Phi\}.$$

Then $\mathcal{M}^{\Phi}$ is a decomposition of the space X. The quotient space with respect to this decomposition will be denoted by Y^{Φ}, the quotient map $X \to Y^{\Phi}$ by f^{Φ}. Since $\mathcal{M}^{\Phi}$ refines the decomposition $\mathcal{M}_f = \{f^{-1}y: y \in Y\}$, there is a unique map $\pi^{\Phi}: Y^{\Phi} \to Y$ such that $f = \pi^{\Phi} f^{\Phi}$.

2.5. Lemma. *The space Y^{Φ} is Hausdorff and the map $f^{\Phi}: X \to Y^{\Phi}$ is fully closed.*

Proof. If $z_1, z_2 \in Y^{\Phi}$ and $\pi^{\Phi} z_1 \neq \pi^{\Phi} z_2$, then $(\pi^{\Phi})^{-1} U_1$ and $(\pi^{\Phi})^{-1} U_2$, where U_1 and U_2 are disjoint neighbourhoods of $\pi^{\Phi} z_1$ and $\pi^{\Phi} z_2$ respectively, will be disjoint neighbourhoods of z_1 and z_2. Now let $\pi^{\Phi} z_1 = \pi^{\Phi} z_2 = y$. Then $(f^{\Phi})^{-1} z_1$ and $(f^{\Phi})^{-1} z_2$ are disjoint compact subsets of X. Let U_1 and U_2 be corresponding disjoint neighbourhoods. We put $V_i = (f^{\Phi})^{\#} U_i^y$, $i = 1, 2$. Since $(f^{\Phi})^{-1} z_i \subset U_i^y$, we have $z_i \in V_i$. Moreover $V_1 \cap V_2 = \emptyset$. It remains to show that V_i is a neighbourhood of z_i, that is $(f^{\Phi})^{-1} V_i$ is open. We have

$$\begin{aligned}(f^{\Phi})^{-1} V_i &= (f^{\Phi})^{-1} (f^{\Phi})^{\#} ((U_i \cap f^{-1} y) \cup f^{-1} f^{\#} U_i) \\ &= ((f^{\Phi})^{-1} (f^{\Phi})^{\#} U_i \cap f^{-1} y) \cup f^{-1} f^{\#} U_i.\end{aligned}$$

So we have to check that every point $x \in (f^{\Phi})^{-1} (f^{\Phi})^{\#} U_i \cap f^{-1} y \equiv {}^y U_i$ is an interior point of $(f^{\Phi})^{-1} V_i$. The set ${}^y U_i$ is open in $f^{-1} y$, since ${}^y U_i = \phi_y^{-1} \phi_y^{\#} (U_i \cap f^{-1} y)$. Hence there is an open set $W_i \subset X$ such that $W_i \cap f^{-1} y = {}^y U_i$. Therefore $W_i \cap U_i^y$ is a neighbourhood of x, which is contained in $(f^{\Phi})^{-1} V_i$. Thus Y^{Φ} is Hausdorff.

To prove that f^{Φ} is fully closed one needs to show that for every open $U \subset X$ and every $z \in Y^{\Phi}$ the set U^z is open. Since Y^{Φ} is Hausdorff, f^{Φ} is closed and so $(f^{\Phi})^{-1} (f^{\Phi})^{\#} U$ is open, since $(f^{\Phi})^{\#} U = Y^{\Phi} \setminus f^{\Phi}(X \setminus U)$. Hence we have to check that every point $x \in U \cap (f^{\Phi})^{-1} z$ is an interior point of U^z. Let $y = \pi^{\Phi} z \equiv fx$, $V = U \cap f^{-1} y$. The map $\Phi_y: f^{-1} y \to (\pi^{\Phi})^{-1} y$ is fully closed, hence by Lemma 2.3 there is an open set $W_1 \subset X$ such that

$$x \in W_1 \cap f^{-1} y \subset (\phi_y^{-1} z \cap V) \cup \phi_y^{-1} \phi_y^{\#} V. \tag{0}$$

Let $W_2 = W_1 \cap U^y$. We shall show that $x \in W_2 \subset U^z$. It is clear that $x \in W_2$, because

$$x \in U \cap (f^{\Phi})^{-1} z \subset U \cap f^{-1} y \subset U^y.$$

Further,

$$W_1 \cap (f^{-1} f^{\#} U) \subset f^{-1} f^{\#} U \subset (f^{\Phi})^{-1} (f^{\Phi})^{\#} U.$$

It remains only to show that $W_1 \cap U \cap f^{-1} y \subset U^z$. In view of (0) it is sufficient to check that

$$(\phi_y^{-1} z \cap V) \cup \phi_y^{-1} \phi_y^{\#} V \subset U^z.$$

But

$$\phi_y^{-1} z \cap V = \phi_y^{-1} z \cap U = (f^{\Phi})^{-1} z \cap U \subset U^z.$$

On the other hand

$$\phi_y^{-1}\phi_y^{\#}V=(f^{\Phi})^{-1}(f^{\Phi})^{\#}(U\cap f^{-1}y)\subset U^z.$$

Lemma 2.5. is proved.

Let $\Phi_y=\{f^{-1}y'\xrightarrow{\phi_{y'}}Z_{y'}: y'\in Y\}$ be a family, such that

$$Z_{y'}=\{y'\}\quad\text{if } y'\neq y,$$
$$Z_y=f^{-1}y\quad\text{and}\quad \phi_y=1_{f^{-1}y}.$$

Then we denote $Y^{\Phi y}, f^{\Phi y}, \pi^{\Phi y}$ by Y^y_f, f^y, π^y_f respectively. Sometimes we shall also use the *notations*: Y^y for Y^y_f and π^y for π^y_f.

Example. If, as in 2.1. above, X is the lexicographic square, Y is $[0,1]$ and y is a chosen point of Y then Y^y can be identified with the subspace $([0,1]\times\{0\})\cup(\{y\}\times[0,1])$ of X (that is, a segment plus a spike with a lexicographic topology). Then f^y and π^y would be the obvious retractions onto Y^y and Y respectively. Thus f^y collapses X to a one-point fattening of Y about y.

From Lemma 2.5 we obtain:

2.6. Corollary [2]. *If $f:X\to Y$ is a fully closed map and $y\in Y$, then the space Y^y is Hausdorff and the map $f^y:X\to Y^y$ is fully closed.*

2.7. Lemma [12]. *If $f:X\to Y$ is a map, such that Y^y is Hausdorff for the point $y\in Y$, then f is fully closed at y.*

2.8. Lemma. *Let $f:X\to Y$ and $g:Y\to Z$ be fully closed maps, such that gf is also fully closed and let $z\in Z$. Then the map $g^zf:X\to Z^z$ is also fully closed.*

Proof. Let $y\in g^{-1}z$. The proof proceeds by reference to some intermediate fattenings about y and z. We first introduce these formally and, before continuing with the proof, illustrate them in a particular case with a view to clarifying the notation. We consider the following diagram

(1)
$$\begin{tikzcd}
Z & & Y \arrow[ll, "g"'] \arrow[dl, "g^z"] & & X \arrow[ll, "f"'] \arrow[dl, "f^y"] \\
& Z^z_g \arrow[ul, "\pi^z_g"] & & Y^y_f \arrow[ul, "\pi^y_f"] &
\end{tikzcd}$$

The map $h\equiv g\pi^y_f$ is fully closed by Lemma 2.2. Now we consider the diagram

(2)
$$\begin{tikzcd}
Z & & Y_f^y \arrow[ll, "h"'] \arrow[dl, "h^z"] \\
& Z_h^z \arrow[ul, "\pi_h^z"] &
\end{tikzcd}$$

There is a unique map $k: Z_h^z \to Z_g^z$ such that $\pi_h^z = \pi_g^z k$. So we have the commutative diagram

(3)
$$\begin{tikzcd}
Z & & Y \arrow[ll, "g"'] \arrow[dl, "g^z"] & & X \arrow[ll, "f"'] \arrow[dl, "f^y"] \\
& Z_g^z \arrow[ul, "\pi_g^z"] & & Y_f^y \arrow[ul, "\pi_f^y"] \arrow[dl, "h^z"] & \\
& & Z_h^z \arrow[ul, "k"] & &
\end{tikzcd}$$

Let us examine this formal diagram when X is the lexicographic cube (ordered by first difference in the three coordinates), Y the lexicographic square and Z the unit interval with f projecting onto the first two coordinates and g, similarly, projecting from Y onto the first coordinate. The formal diagram above may be replaced by the self-explanatory picture (see Fig. 1.). Now we are ready to prove that $g^z f$ is fully closed. If $x \in Z_g^z$ and $|(\pi_g^z)^{-1}\pi_g^z x| = 1$, then $g^z f$ is fully closed at x by Lemma 2.4. Now let $|(\pi_g^z)^{-1}\pi_g^z x| > 1$. Then $(g^z)^{-1}x \equiv y \in g^{-1}z$. Let $\{U_i : i = 1, \ldots, s\}$ be a finite covering of the set $(g^z f)^{-1}x = f^{-1}y$ by sets open in X. By definition $f^y \,|\, f^{-1}y$ is a one-to-one correspondence. Consequently the open family $\{(f^y)^{\#} U_i : i = 1, \ldots, s\}$ is a covering of the set $(\pi_f^y)^{-1}y = (g^z\pi_f^y)^{-1}x = (kh^z)^{-1}x$. Further, $(\pi_f^y)^{-1}y \subset h^{-1}z$, so $h^z \,|\, (\pi_f^y)^{-1}y$ is a one-to-one correspondence. Therefore the open family $\{(h^z)^{\#}(f^y)^{\#} U_i : i = 1, \ldots, s\}$ is a covering of the set $h^z(\pi_f^y)^{-1}y = k^{-1}x$. The map k is fully closed, since it can have only one non-trivial inverse image $k^{-1}x$. Hence $\{x\} \cup (\bigcup_{i=1}^{s} k^{\#}(h^z)^{\#}(f^y)^{\#} U_i)$ is open But $kh^z f^y = g^z f$ (look at diagram (3)) and $k^{\#}(h^z)^{\#}(f^y)^{\#} = (kh^z f^y)^{\#}$. Thus $g^z f$ is fully closed at x. Lemma 2.8 is proved.

2.9. Lemma. *Let $S = \{X_\alpha, \pi_\alpha^{\alpha'} : \alpha, \alpha' \in A\}$ and $S' = \{Y_\alpha, \rho_\alpha^{\alpha'} : \alpha, \alpha' \in A\}$ be inverse spectra on the same directed set A, which satisfies the following condition: for each countable set $B \subset A$ there is an $\alpha \in A$ such that $\alpha \geqslant \beta$ for all $\beta \in B$ (for example, a well-ordered set A with* cf $A \geqslant \omega_1$ *has this property). If $F: S \to S'$ is a morphism, which consists of fully closed maps, then* $\lim F$: $\lim S \to \lim S'$ *is also fully closed.*

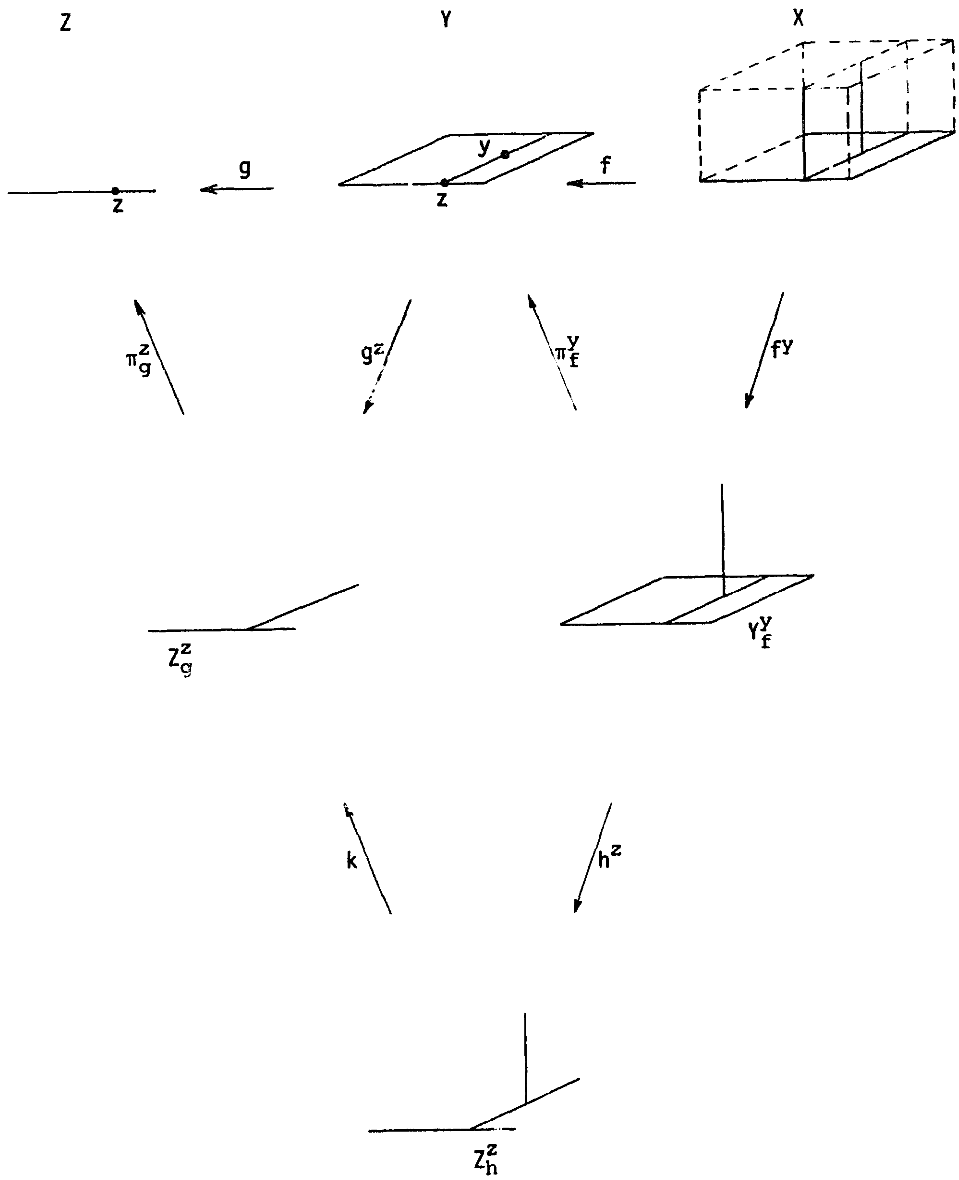

Fig. 1.

Proof. It follows from Definition 2.1 that a map $f: X \to Y$ is fully closed if and only if for any finite open covering $\{U_i: i = 1, \dots, s\}$ of X the set $Y \backslash \bigcup_{i=1}^{s} f^{\#} U_i$ is discrete and so finite. Let $\lim F = f$, $\lim S' = Y$ and let $\{U_i: i = 1, \dots, s\}$ be an open covering of $X = \lim S$. We suppose that the set $\Phi = Y \backslash \bigcup_{i=1}^{s} f^{\#} U_i$ is infinite. By the hypothesis on A, for some $\alpha \in A$ the set $\rho_\alpha \Phi$ is infinite. Hence $\rho_\beta \Phi$ is infinite for all $\beta \geqslant \alpha$. But

$$\rho_\beta \Phi = \rho_\beta \Big(Y \backslash \bigcup_{i=1}^{s} f^{\#} U_i\Big)$$

$$= Y_\beta \backslash \rho_\beta^{\#} \bigcup_{i=1}^{s} f^{\#} U_i \subset Y_\beta \backslash \bigcup_{i=1}^{s} \rho_\beta^{\#} f^{\#} U_i = Y_\beta \backslash \bigcup_{i=1}^{s} f_\beta^{\#} \pi_\beta^{\#} U_i.$$

Now we can take $\beta \geq \alpha$ such that $\{\pi_\beta^\# U_i: i = 1, \ldots, s\}$ is a covering of X_β. Since $Y_\beta \setminus \bigcup_{i=1}^s f_\beta^\# \pi_\beta^\# U_i$ is infinite and f_β is fully closed we have a contradiction. Lemma 2.9 is proved.

2.10. Example. There exists a morphism $F: S \to S'$ from a countable inverse spectrum $S = \{X_n, \pi_n^m: m, n \in \omega\}$ to a countable inverse spectrum $S' = \{Y_n, \rho_n^m: m, n \in \omega\}$ such that all $f_n \in F$ are fully closed, but $\lim F$ is not fully closed.

The spaces $Y_n = \{y_1, \ldots, y_n\}$ and $X_n = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ are finite. The map π_n^{n+1} sends x_{n+1} to x_n and y_{n+1} to y_n and the map ρ_n^{n+1} sends y_{n+1} to y_n and are identity elsewhere; finally $f_n^{-1} y_i = \{x_i, y_i\}$, $i = 1, \ldots, n$. It is evident that $\lim S$ is the discrete sum of two convergent sequences $C_1 = \{x_1, \ldots, x_n, \ldots, x_\infty\}$ and $C_2 = \{y_1, \ldots, y_n, \ldots, y_\infty\}$; $\lim S'$ is the convergent sequence C_2; the map $\lim F$ is an identification of C_1 and C_2. All maps f_n are fully closed, since all Y_n are finite. But $\lim F$ is not fully closed at the point y_∞.

2.11. Lemma [2]. *Let $S = \{X_\alpha, \pi_\alpha^{\alpha'}: \alpha, \alpha' \in A\}$ be an inverse spectrum and let $\beta \in A$. If all maps π_β^α are fully closed, then the limit projection $\pi_\beta: \lim S \to X_\beta$ is also fully closed.*

2.12. Definition. We shall say that the topology of a space X is openly generated on the set $F \subset X$ by a family $\{X \xrightarrow{f_\alpha} X_\alpha: \alpha \in A\}$, if for every point $x \in F$ and every neighbourhood U of x there exist an α and a neighbourhood V of the point $f_\alpha x$ such that $f_\alpha^{-1} V \subset U$.

2.13. Lemma. *Let $f: X \to Y$ be a map, such that there are maps $f_\alpha: X \to X_\alpha$ and $\pi_\alpha: X_\alpha \to Y$, $\alpha \in A$, satisfying*

(1) *$\pi_\alpha f_\alpha = f$ for all $\alpha \in A$;*

(2) *π_α is fully closed at a point $y \in Y$ for every $\alpha \in A$;*

(3) *the topology of X is openly generated on the set $f^{-1} y$ by the family $\{X \xrightarrow{f_\alpha} X_\alpha: \alpha \in A\}$.*

Then f is fully closed at y.

Proof. By Lemma 2.3 it is sufficient to prove for every set U, open in X, the set U^y is also open. One needs only to check that each $x \in U \cap f^{-1} y$ is an interior point of U^y. There exist an $\alpha \in A$ and an open set $V \subset X_\alpha$ such that $x \in f_\alpha^{-1} V \subset U$. Since π_α is fully closed at y, the set V^y is open. Then $x \in f_\alpha^{-1} V^y \subset U^y$. Indeed,

$$x \in f_\alpha^{-1} V \cap f^{-1} y = f_\alpha^{-1} (V \cap \pi_\alpha^{-1} y) \subset f_\alpha^{-1} V^y.$$

On the other hand

$$\begin{aligned} f_\alpha^{-1} V^y &= f_\alpha^{-1}((V \cap \pi_\alpha^{-1} y) \cup \pi_\alpha^{-1} \pi_\alpha^\# V) \\ &= (f_\alpha^{-1} V \cap f^{-1} y) \cup f^{-1} \pi_\alpha^\# V \subset (U \cap f^{-1} y) \cup f^{-1} f^\# U, \end{aligned}$$

because of $f_\alpha^{-1} V \subset U$ and $\pi_\alpha^\# V \subset f^\# U$. Lemma 2.13 is proved.

2.14. Lemma. *If $f: X \to Y$ is fully closed and F is closed in X, then $g = f|F : F \to fF$ is also fully closed.*

Proof. Because of Lemma 2.3, one needs only to check that every set U, open in F, and every point $y \in fF$ the set $U^y = (U \cap g^{-1}y) \cup g^{-1}g^{\#}U$ is open in F. Let V be an open subset of X such that $U = F \cap V$. The set $V^y = (f^{-1}y \cap V) \cup f^{-1}f^{\#}V$ is open in X by Lemma 2.3. Further,

$$V^y \cap F = (g^{-1}y \cap U) \cup (F \cap f^{-1}f^{\#}V).$$

But $F \cap f^{-1}f^{\#}V \subset g^{-1}g^{\#}U$. Indeed, if $x \in F \cap f^{-1}f^{\#}V$, then $f^{-1}fx \subset V$, so

$$g^{-1}gx = F \cap f^{-1}fx \subset F \cap V \subset U.$$

Therefore $U \cap g^{-1}y \subset V^y \cap F \subset U^y$. Thus U^y is open in F, being the union of the two open sets $V^y \cap F$ and $g^{-1}g^{\#}U$. Lemma 2.14 is proved.

3. Roll of a spectral tree

Let P be a tree, that is a partially ordered set with a least element such that for each $X \in P$ the set $(-\infty, X)$ is well-ordered. The order type of this set we shall call the height of the element X and we shall denote it by $h(X)$. The least ordinal $\alpha \equiv h(P)$, such that $h(X) < \alpha$ for each $X \in P$, will be called the height of P.

Let ξ be a maximal chain of P and let α be less than the order type of ξ. Then there is a unique element $X \in P$ such that $X \in \xi$ and $h(X) = \alpha$. Sometimes it will be more convenient for us to denote this element X by X^{ξ}_{α}. If $X \in P$ and $h(X)$ is an isolated ordinal, then the unique predecessor of X will be denoted by ^{-}X.

3.1. Definition. Let P be a tree, whose objects are topological spaces. A set $T = \{X, \pi^Y_X: X, Y \in P\}$ where $\pi^Y_X: Y \to X$ is a map for each pair $X, Y \in P$ such that $X \leq Y$, will be called a spectral tree if $\pi^X_X = 1_X$ and $\pi^Y_X \pi^Z_Y = \pi^Z_X$ for any three elements $X, Y, Z \in P$ such that $X \leq Y \leq Z$.

For given $\alpha \leq h(T) \equiv h(P)$ we shall denote the set $\{X, \pi^Y_X: h(X), h(Y) < \alpha\}$ by $T|\alpha$. Clearly $T|\alpha$ is a spectral tree of height α. If $X \in T$, then $T|X \equiv \{Y, \pi^Z_Y: Y, Z < X\}$. Clearly $T|X$ is an inverse spectrum for each $X \in T$.

3.2. Definition. A spectral tree T is called *continuous*, if for each $X \in T$, whose height is a limit ordinal, the natural projection from X to $\lim (T|X)$ is a homeomorphism. In addition we require that the least element of T consists of one point.

3.3. Definition. The family $\{X \xleftarrow{} Y \xrightarrow{h_\alpha} Y_\alpha : \alpha \in A\}$ is called *a fan-product* of the family

$$\{X \xrightarrow{f_\alpha} X_\alpha \xleftarrow{g_\alpha} Y_\alpha : \alpha \in A\}$$

if each diagram

$$\begin{tikzcd} X \arrow[d, "f_\alpha"'] & Y \arrow[l, "g"'] \arrow[d, "h_\alpha"] \\ X_\alpha & Y_\alpha \arrow[l, "g_\alpha"] \end{tikzcd}$$

is commutative and for any family $\{X \xleftarrow{\hat{g}} \hat{Y} \xrightarrow{\hat{h}_\alpha} Y_\alpha : \alpha \in A\}$ such that each diagram

$$\begin{tikzcd} X \arrow[d, "f_\alpha"'] & \hat{Y} \arrow[l, "\hat{g}"'] \arrow[d, "\hat{h}_\alpha"] \\ X_\alpha & Y_\alpha \arrow[l, "g_\alpha"] \end{tikzcd}$$

is commutative, there exists a unique map $h : \hat{Y} \to Y$ such that $\hat{g} = gh$ and $\hat{h}_\alpha = h_\alpha h$ for each $\alpha \in A$.

3.4. Theorem. *Each family* $\{X \xrightarrow{f_\alpha} X_\alpha \xleftarrow{g_\alpha} Y_\alpha : \alpha \in A\}$ *has a unique fan-product.*

Proof. Firstly *uniqueness*. Let

$$\{X \xleftarrow{g} Y \xrightarrow{h_\alpha} Y_\alpha\} \quad \text{and} \quad \{X \xleftarrow{\hat{g}} \hat{Y} \xrightarrow{\hat{h}_\alpha} Y_\alpha\}$$

be the fan-products. Let $h : \hat{Y} \to Y$ and $h' : Y \to \hat{Y}$ be the unique maps, which satisfy the definition of a fan-product. Then $ghh' = g$ and $h_\alpha hh' = h_\alpha$ for each $\alpha \in A$. But according to the definition of a fan-product there exists a unique map $k : Y \to Y$ such that $gk = g$ and $h_\alpha k = h_\alpha$ for each $\alpha \in A$. Therefore $hh' = 1_Y$. In the same way we show, that $h'h = 1_{\hat{Y}}$. Hence the fan-product is unique.

Now *existence*. First of all let

$$\begin{tikzcd} X \arrow[d, "f_\alpha"'] & Z_\alpha \arrow[l, "p_\alpha"'] \arrow[d, "q_\alpha"] \\ X_\alpha & Y_\alpha \arrow[l, "g_\alpha"] \end{tikzcd}$$

be a pullback square. Then let $\{Y \xrightarrow{\tau_\alpha} Z_\alpha : \alpha \in A\}$ be a fibre-product (or generalized pullback) of the family $\{Z_\alpha \xrightarrow{p_\alpha} X : \alpha \in A\}$. Finally let $h_\alpha = q_\alpha \tau_\alpha$ and $g = p_\alpha \tau_\alpha$. Clearly h_α and g are epimorphisms. It follows from the definition that $f_\alpha g = f_\alpha p_\alpha \tau_\alpha = g_\alpha q_\alpha \tau_\alpha = g_\alpha h_\alpha$. We observe that

$$Y = \{(x, \langle y_\alpha : \alpha \in A\rangle) : f_\alpha(x) = g_\alpha(y_\alpha)\} \subset X \times \prod Y_\alpha.$$

Let

$$\{X \xleftarrow{\hat{g}} \hat{Y} \xrightarrow{\hat{h}_\alpha} Y_\alpha : \alpha \in A\}$$

be an arbitrary family such that $f_\alpha \hat{g} = g_\alpha \hat{h}_\alpha$ for each $\alpha \in A$. It follows from the definition of a pullback square that for every $\alpha \in A$ there exists a unique map $\hat{\tau}_\alpha : \hat{Y} \to Z_\alpha$ such that $p_\alpha \hat{\tau}_\alpha = \hat{g}$ and $q_\alpha \hat{\tau}_\alpha = \hat{h}_\alpha$. Further, according to the definition of a generalized pullback there exists a unique map $h : \hat{Y} \to Y$ such that $\tau_\alpha h = \hat{\tau}_\alpha$. Therefore

$$\hat{g} = p_\alpha \hat{\tau}_\alpha = p_\alpha \tau_\alpha h = gh$$

and

$$\hat{h}_\alpha = q_\alpha \hat{\tau}_\alpha = q_\alpha \tau_\alpha h = h_\alpha h.$$

Theorem 3.4 is proved.

3.5. Lemma. *Let $\{X \xleftarrow{g} Y \xrightarrow{h_\alpha} Y_\alpha : \alpha \in A\}$ be the fan-product of the family*

$$\{X \xrightarrow{f_\alpha} X_\alpha \xleftarrow{g_\alpha} Y_\alpha : \alpha \in A\}$$

and let $x \in X$. Then $g^{-1}x$ is homeomorphic to the product $\prod \{g_\alpha^{-1} f_\alpha x : \alpha \in A\}$.

Proof. We follow the proof of Theorem 3.4. According to the definition of a fibre-product $p_\alpha^{-1}x$ is homeomorphic to $g_\alpha^{-1} f_\alpha x$. Applying the definition of a fibre-product once more we get $g^{-1}x \approx \prod p_\alpha^{-1}x \approx \prod g_\alpha^{-1} f_\alpha x$. Lemma 3.5 is proved.

Let T be a continuous spectral tree of height α. A family $R = \{S, \pi_X : X \in T\}$, where $S = \{Y_\beta, \pi_\beta^{\beta'} : \beta, \beta' < \alpha\}$ is a continuous spectrum and $\pi_X : Y_{h(X)} \to X$ is a map onto X, will be called a *cover* of the spectral tree T, if $\pi_X \pi_{h(X)}^{h(X')} = \pi_X^{X'} \pi_{X'}$, for all X, $X' \in T$ such that $X \leqslant X'$.

Let $\beta < h(T)$ and let R be a cover of $T \mid \beta$. A cover R' of T is called an extension of a cover R on T, if $R' \mid \beta = R$.

3.6. Lemma. *Let T be a continuous spectral tree of height α and let $\beta < \alpha$. Then for every cover R of $T \mid \beta$ there exists its extension R' on T.*

Proof. A cover R' is constructed by induction. If β is a limit ordinal, we use the passage to the limit of a spectrum. If β is isolated, then

$$\{Y_{\beta-1} \xleftarrow{\pi_{\beta-1}^{\beta}} Y_\beta \xrightarrow{\pi_X} X : X \in T, h(X) = \beta\}$$

is the fan-product of the family

$$\{Y_{\beta-1} \xrightarrow{\pi_{-X}} {}^{-}X \xleftarrow{\pi_{-X}^{X}} X : X \in T, h(X) = \beta\}.$$

3.7. Definition. We shall say that a cover R of a spectral tree T is a *roll* of T, if for any cover $\hat{R} = \{\hat{S}, \hat{\pi}_X : X \in T\}$ there exists a unique morphism $F = \{f_\beta\} : \hat{S} \to S$ such that $\hat{\pi}_X = \pi_X f_{h(X)}$ for all $X \in T$.

Now by transfinite recursion we construct a cover $\mathcal{R}(T) = \{S, \pi_X : X \in T\}$ of T, where $S = \{Y_\alpha, \pi_\alpha^{\alpha'} : \alpha, \alpha' < h(T)\}$, which satisfies the following property: the family

$$\{Y_\alpha \xleftarrow{\pi_\alpha^{\alpha+1}} Y_{\alpha+1} \xrightarrow{\pi_X} X : X \in T, h(X) = \alpha + 1\}$$

is the fan-product of

$$\{Y_\alpha \xrightarrow{\pi_{-X}} {}^-X \xleftarrow{\pi_X^-} X : X \in T, h(X) = \alpha + 1\}.$$

3.8. Properties of $\mathcal{R}(T)$. (1) $\mathcal{R}(T|\alpha) = \mathcal{R}(T)|\alpha$.
If $h(T)$ is a limit ordinal, then
(2) $\mathcal{R}(T) = \bigcup\{\mathcal{R}(T|\alpha) : \alpha < h(T)\}$.
(3) $\overline{\mathcal{R}(T)} = \mathcal{R}(\bar{T})$.

Here for an arbitrary category $\mathcal{A}$ we denote by $\bar{\mathcal{A}}$ the completion of $\mathcal{A}$ by all limit spaces and all limit projections.

3.9. Theorem. *The cover $\mathcal{R}(T)$ is the unique roll of T.*

Proof. Uniqueness follows from the definition of a roll. We show that $\mathcal{R}(T)$ is a roll of T by induction on the height of T. If $h(T) = 1$, then $T = \mathcal{R}(T)$ is a roll of itself. We suppose that for every continuous spectral tree T of height $\alpha' < \alpha$ the cover $\mathcal{R}(T)$ is a roll of T.

Let T be a continuous spectral tree of height α and let $R = \{S', \rho_X : X \in T\}$ be an arbitrary cover of T. Since $\mathcal{R}(T)|\gamma = \mathcal{R}(T|\gamma)$, there exists a unique morphism

$$F_\gamma = \{f_\beta^\gamma : \beta < \gamma\} : S'|\gamma \to S|\gamma$$

such that $\pi_X f_{h(X)}^\gamma = \rho_X$ for all $X \in T|\gamma$. Because of uniqueness of F_γ, the sequence $\{F_\gamma : \gamma < \alpha\}$ is increasing and so $f_\beta^\gamma = f_\beta^{\gamma+1} = \cdots = f_\beta$.

If α is a limit ordinal, then $F = \{f_\beta\} : S' \to S$ is the unique morphism such that $\pi_X f_{h(X)} = \rho_X$ for all $X \in T$. Hence $\mathcal{R}(T)$ is a roll of T.

Now let $\alpha = (\alpha - 1) + 1$, where $\alpha - 1$ is a limit ordinal. It follows from the continuity of the spectra S' and S, that the morphism $F_{\alpha-1} : S'|\alpha - 1 \to S|\alpha - 1$ has a unique extension to $F : S' \to S$. Moreover $\pi_X f_{\alpha-1} = \rho_X$ for all $X \in T$ such that $h(X) = \alpha - 1$, because of continuity of T. So $\mathcal{R}(T)$ is a roll of T.

Finally let $\alpha = (\alpha - 2) + 2$. It follows from the definition of a fan-product that there is a unique map $f_{\alpha-1} : Z_{\alpha-1} \to Y_{\alpha-1}$, where $S' = \{Z_\beta, \rho_\beta^{\beta'} : \beta, \beta' < \alpha\}$ such that $\pi_{\alpha-2}^{\alpha-1} f_{\alpha-1} = f_{\alpha-2} \rho_{\alpha-2}^{\alpha-1}$ and $\pi_X f_{\alpha-1} = \rho_X$ for all $X \in T$, $h(X) = \alpha - 1$. Therefore again there exists a unique extension of $F_{\alpha-1} : S'|\alpha - 1 \to S|\alpha - 1$ to $F : S' \to S$ such that $\pi_X f_{\alpha-1} = \rho_X$. Thus $\mathcal{R}(T)$ is a roll of T. Theorem 3.9 is proved.

3.10. Remark. If $\mathcal{R}(T)=\{S, \pi_X: X\in T\}$, then the continuous spectrum $S=S(T)$ will also be referred to as the roll of the spectral tree T.

4. Scannable spectra and their scannings

4.1. Definition. A continuous spectrum $S=\{X_\alpha, \pi_\alpha^{\alpha'}: \alpha, \alpha'<\beta\}$ is called *simple* if

(1) for every α, such that $\alpha+1<\beta$, there is a distinguished point $\bar{x}_\alpha\in X_\alpha$ such that $|(\pi_\alpha^{\alpha+1})^{-1}x|=1$ for all $x\in X_\alpha\backslash\{\bar{x}_\alpha\}$;

(2) $\pi_\alpha^{\alpha'}\bar{x}_{\alpha'}=\bar{x}_\alpha$.

Now let $S=\{Y_\alpha, \rho_\alpha^{\alpha'}: \alpha, \alpha'<\beta\}$ and $S'=\{X_\alpha, \pi_\alpha^{\alpha'}: \alpha, \alpha'<\beta\}$ be continuous spectra of the same length. We suppose in addition that S' is a simple spectrum.

4.2. Definition. We shall say that the spectrum S *dominates* S' if there is a morphism $F=\{d_\alpha\}: S\to S'$; which will be called a *domination*, such that

(a) d_α is a map onto,

(b) $|d_\alpha^{-1}\bar{x}_\alpha|=1$,

(c) $d_{\alpha+1}|(\rho_\alpha^{\alpha+1})^{-1}d_\alpha^{-1}\bar{x}_\alpha$ is a homeomorphism.

The family $\xi=\{d_\alpha^{-1}\bar{x}_\alpha: \alpha+1<\beta\}$ is a *thread* of the spectrum ${}^-S=\{Y_\alpha, \rho_\alpha^{\alpha'}: \alpha+1, \alpha'+1<\beta\}$, that is $\xi\in\lim {}^-S\equiv\lim_0 S$. We shall say that S dominates S' with respect to ξ.

4.3. Lemma. *If $F: S\to S'$ is a domination with respect to ξ, then $F|\alpha: S|\alpha\to S'|\alpha$ is a domination with respect to ${}_0\pi_\alpha\xi$, where ${}_0\pi_\alpha: \lim_0 S\to\lim_0(S|\alpha)$ is the natural projection, for every $\alpha<l(S)$.*

Proof is trivial.

4.4. Lemma. *A domination $F: S\to S'$ with respect to $\xi\in\lim_0 S$ is unique.*

Proof. Let $S=\{Y_\alpha, \rho_\alpha^{\alpha'}: \alpha, \alpha'<\beta\}$, $F=\{d_\alpha\}$ and $S'=\{X_\alpha, \pi_\alpha^{\alpha'}: \alpha, \alpha'<\beta\}$. We consider the diagram

$$\begin{tikzcd} Y_\alpha \arrow[d, "d_\alpha"'] & Y_{\alpha+1} \arrow[l, "\rho_\alpha^{\alpha+1}"'] \arrow[d, "d_{\alpha+1}"] \\ X_\alpha & X_{\alpha+1} \arrow[l, "\pi_\alpha^{\alpha+1}"'] \end{tikzcd}$$

and put $X=Y_{\alpha+1}$, $Y=Y_\alpha$, $f=d_\alpha\rho_\alpha^{\alpha+1}$ and $y=\bar{x}_\alpha$. Then conditions (1), (2) and (a), (b), (c) from Definitions 4.1 and 4.2 imply that $X_{\alpha+1}=Y^y$, $d_{\alpha+1}=f^y$ and $\pi_\alpha^{\alpha+1}=\pi^y$. (One has only to note that the decompositions of $Y_{\alpha+1}$ induced by $d_{\alpha+1}$ and f^y are identical.) Together with continuity of S', this gives us the uniqueness of F. Lemma 4.4 is proved.

4.5. Definition. Let S be a continuous spectrum. If for every $\xi \in \lim_0 S$ there exists a unique domination $F_\xi: S \to S_\xi$ with respect to ξ, then the spectrum S is called *scannable*.

From Lemmas 4.3 and 4.4 we get

4.6. Lemma. *If S is a scannable spectrum of length β, then for every $\alpha < \beta$ the spectrum $S|\alpha$ is also scannable.*

Let $S = \{X_\alpha, \pi_\alpha^{\alpha'}: \alpha, \alpha' < \beta\}$ be a scannable spectrum of length β. We put $L_\alpha(S) = \lim_0(S|\alpha+1)$ for each $\alpha < \beta$. We shall denote by ${}^0\pi_\alpha^{\alpha'}$ the natural projection $L_{\alpha'}(S) \to L_\alpha(S)$. The projection $\lim_0 S \to L_\alpha(S)$ will be denoted by ${}^0\pi_\alpha$. So we have ${}^0\pi_\alpha = {}_0\pi_{\alpha+1}$.

If α is a limit ordinal, then $L_\alpha(S) = X_\alpha$, and if $\alpha = (\alpha-1)+1$, then $L_\alpha(S) = X_{\alpha-1}$.

Let $x \in L_\alpha(S)$, $y \in L_{\alpha'}(S)$. We shall say that $x \leq y$ if and only if $\alpha \leq \alpha'$ and $x = {}^0\pi_\alpha^{\alpha'} y$. With respect to this ordering the set $L(S) = \bigcup\{L_\alpha(S): \alpha < \beta\}$ is a tree.

Now let $s \in L_\alpha(S)$ and let

$$F_{\xi_i} = \{d_{\alpha'}^{\xi_i}\}: S \to S_{\xi_i} = \{X_{\alpha'}^{\xi_i}, {}^{\xi_i}\pi_{\alpha'}^{\alpha''}: \alpha', \alpha'' < \beta\}$$

be the dominations such that ${}^0\pi_\alpha\xi_1 = {}^0\pi_\alpha\xi_2 = s$. By Lemma 4.3 $F_{\xi_i}|\alpha+1: S|\alpha+1 \to S_{\xi_i}|\alpha+1$ is a domination with respect to ${}_0\pi_{\alpha+1}\xi_i$. But ${}_0\pi_{\alpha+1} = {}^0\pi_\alpha$. So from the uniqueness of a domination we get $S_{\xi_1}|\alpha+1 = S_{\xi_2}|\alpha+1$ and $F_{\xi_1}|\alpha+1 = F_{\xi_2}|\alpha+1$. In particular $X_\alpha^{\xi_1} = X_\alpha^{\xi_2}$ and $d_\alpha^{\xi_1} = d_\alpha^{\xi_2}$. Sometimes we shall denote X_α^ξ by X^s or X_α^s and d_α^ξ by d^s or d_α^s, where $s = {}^0\pi_\alpha\xi$.

We consider the set $T(S) = \bigcup\{S_\xi: \xi \in \lim_0 S\}$. If ${}^0\pi_\alpha\xi = {}^0\pi_\alpha\zeta$, then $X_{\alpha'}^\xi = X_{\alpha'}^\zeta$ and ${}^\xi\pi_{\alpha'}^{\alpha''} = {}^\zeta\pi_{\alpha'}^{\alpha''}$ for all $\alpha', \alpha'' \leq \alpha$. With respect to this identification the set $T(S)$ is a spectral tree. Furthermore, if $T(S) = \{X, \pi_X^Y: X, Y \in P\}$, then the tree P is isomorphic to $L(S)$.

4.7. Definition. The spectral tree $T(S)$ will be called *the scanning of the spectrum S*.

A continuous spectral tree $T = \{X, \pi_X^Y: X, Y \in P\}$ will be called a *spectral tree with distinguished points*, if for each pair $X, Y \in P$ such that $X < Y$ there is a distinguished point $x_{(X,Y)} \in X$ such that

(*i*) *if* $X < Y < Z$, then $\pi_X^Y x_{(Y,Z)} = x_{(X,Z)} = x_{(X,Y)}$.

4.8. Definition. A spectral tree T with distinguished points will be called *simple* if

(ii) $x_{(X,Y)} \neq x_{(X,Z)}$ for all X, Y, Z such that $X =^- Y =^- Z$ and $Y \neq Z$.

(iii) $|(\pi_X^Y)^{-1}x| = 1$ for each $x \in X \setminus \{x_{(X,Y)}\}$.

Clearly every simple spectrum is a simple spectral tree.

4.9. Lemma. *If T is a simple spectra tree, $X \in T$, $h(X)$ is a limit ordinal and X has a successor in T, then there is a unique distinguished point in X and so X has precisely one successor in T.*

Proof. Suppose there are at least two distinguished points $x_{(X,Y)}$ and $x_{(X,Z)}$. Since T is continuous, there is a space $U \in T$ such that $U < X$ and $\pi_U^X x_{(X,Z)} \neq \pi_U^X x_{(X,Y)}$. On the other hand from the condition (i) we get $\pi_U^X x_{(X,Z)} = x_{(U,X)} = \pi_U^X x_{(X,Y)}$. This contradiction proves the lemma.

4.10. Definition. We shall say that a simple spectral tree $T = \{X, \pi_X^Y : X, Y \in P\}$ is *saturated*, if

(iv) all maximal chains of P have the same height as P.

(v) the set $(\pi_{-X}^X)^{-1} x_{(-X,X)}$ consists of distinguished points for each $X \in T$ with non-limit height $h(X)$.

4.11. Lemma. *If $S = \{X_\alpha, \pi_\alpha^{\alpha'} : \alpha, \alpha' < \beta\}$ is a scannable spectrum, then its scanning $T(S)$ is a saturated spectral tree.*

Proof. We define the distinguished points. Let $X, Y \in T(S)$ and $X < Y$. There is $\xi \in \lim_0 S$ such that $X = X_\alpha^\xi$ and $Y = X_{\alpha'}^\xi$ from some $\alpha, \alpha' < \beta$. The spectrum S_ξ is simple. Hence there is a unique distinguished point $\bar{x}_\alpha^\xi \in X_\alpha^\xi$. We put $x_{(X,Y)} = \bar{x}_\alpha^\xi$. One has to check that this definition is valid. Let ζ be another point from $\lim_0 S$ such that $X = X_\alpha^\zeta$ and $Y = X_{\alpha'}^\zeta$. We have ${}^0\pi_{\alpha'}\xi = {}^0\pi_{\alpha'}\zeta$ and so ${}^0\pi_{\alpha+1}\xi = {}^0\pi_{\alpha+1}\zeta = x \in X_\alpha$. But $\bar{x}_\alpha^\xi = d_\alpha^\xi x$ by the definition of a domination. Thus $d_\alpha^\xi = d_\alpha^\zeta$ implies $\bar{x}_\alpha^\xi = \bar{x}_\alpha^\zeta$.

This definition of the distinguished points automatically satisfies the condition (i).

Now let $\xi, \zeta \in \lim_0 S$, $X_\alpha^\xi = X_\alpha^\zeta$ and $X_{\alpha+1}^\xi \neq X_{\alpha+1}^\zeta$. This means that ${}^0\pi_\alpha\xi = {}^0\pi_\alpha\zeta$, but ${}^0\pi_{\alpha+1}\xi \neq {}^0\pi_{\alpha+1}\zeta$. Hence

$$\bar{x}_\alpha^\xi = d_\alpha^\xi\, {}^0\pi_{\alpha+1}\xi \neq d_\alpha^\xi\, {}^0\pi_{\alpha+1}\zeta = d_\alpha^\zeta\, {}^0\pi_{\alpha+1}\zeta = \bar{x}_\alpha^\zeta$$

and so the condition (ii) is satisfied.

The simplicity of the spectrum S_ξ gives us the condition (iii). The condition (iv) is trivial. Finally the condition (c) from Definition 4.2 implies the condition (v). Lemma 4.11 is proved.

4.12. Theorem. *If S is a scannable spectrum, then the family $R = \{S, d^s : s \in L(S)\}$ is the roll of the spectral tree $T(S)$.*

Proof. Firstly we remark that R is a cover of $T(S)$. Now let $S = \{X_\alpha, \pi_\alpha^{\alpha'} : \alpha, \alpha' < \beta\}$ and let $\hat{R} = \{\hat{S}, \hat{\pi}_s : s \in L(S)\}$, with $\hat{S} = \{\hat{X}_\alpha, \hat{\pi}_\alpha^{\alpha'} : \alpha, \alpha' < \beta\}$, be an arbitrary cover of $T(S)$. We have to prove that there exists a unique morphism $F = \{f_\alpha\} : \hat{S} \to S$ such that $\pi_s = d^s f_\alpha$ for all $s \in L_\alpha(S)$, $\alpha < \beta$. Induction on the length of S. If $l(S) = 1$, then the statement is trivial: S and $T(S)$ contain only one space, consisting of one point. Since $S \mid \alpha$ is a scannable tree and $T(S \mid \alpha) = T(S) \mid \alpha$ for all $\alpha < \beta$, we can make an inductive passage to the limit ordinal β. The continuity of spectra S, $\hat{S}$ and the spectral tree $T(S)$ makes it possible to pass from $\beta-1$ to β, where $\beta-1$ is limit.

Now let $\beta=(\beta-2)+2$. One needs to show that there exists a unique map $f_{\beta-1}$ such that the diagram

(1)
$$\begin{tikzcd}
\hat{X}_{\beta-2} \arrow[d, "f_{\beta-2}"'] & \hat{X}_{\beta-1} \arrow[l, "\hat{\pi}^{\beta-1}_{\beta-2}"'] \arrow[d, dashed, "f_{\beta-1}"] \arrow[dr, "\hat{\pi}_s"] & \\
X_{\beta-2} & X_{\beta-1} \arrow[l, "\pi^{\beta-1}_{\beta-2}"'] \arrow[r, "d^s"] & X^s
\end{tikzcd}$$

is commutative for all $s\in L_{\beta-1}(S)$.

Let $z\in\hat{X}_{\beta-1}$ and let $y=f_{\beta-2}\hat{\pi}^{\beta-1}_{\beta-2}z\in X_{\beta-2}$. Since $X_{\beta-2}=L_{\beta-1}(S)$, we have $y\in L_{\beta-1}(S)$. Let $u\in L_{\beta-2}(S)$ be the predecessor of y. We consider the diagram

$$\begin{tikzcd}
X_{\beta-2} \arrow[d, dashed, "d^u"'] & X_{\beta-1} \arrow[l] \arrow[d, "d^y"] \\
X^u & X^y \arrow[l, "\pi^y_u"']
\end{tikzcd}$$

The point $v=d^u y$ is distinguished in X^u and $d^y\colon(\pi^{\beta-1}_{\beta-2})^{-1}y\to(\pi^y_u)^{-1}v$ is a one-to-one correspondence by property (c) from Definition 4.2. So the equation $\hat{\pi}_y z=d^y f_{\beta-1}z$ defines $f_{\beta-1}$ uniquely and gives the equality $\pi^{\beta-1}_{\beta-2}f_{\beta-1}=f_{\beta-2}\hat{\pi}^{\beta-1}_{\beta-2}$. Now we show that $\hat{\pi}_s z=d^s f_{\beta-1}z$ for every $s\in L_{\beta-1}(S)$ such that $s\neq y$. Let $t\in L_{\beta-2}$ be the predecessor of s. According to the property (b) from Definition 4.2 the point $d^t y$ is not distinguished, so $(\pi^s_t)^{-1}d^t y$ consists of one point. On the other hand $d^s f_{\beta-1}z\in(\pi^s_t)^{-1}d^t y$ and $\hat{\pi}_s z\in(\pi^s_t)^{-1}d^t y$. So $\hat{\pi}_s z=d^s f_{\beta-1}z$.

Finally $f_{\beta-1}$ is continuous because all $d^s f_{\beta-1}$ are continuous and the topology of $X_{\beta-1}$ is openly generated by the family

$$\{X^s \xleftarrow{d^s} X_{\beta-1}\colon s\in L_{\beta-1}(S)\}$$

in view of the property (c) from Definition 4.2 and the closedness of d^s. By Lemma 2.4, Theorem 4.12 follows.

4.13. Lemma. *Every spectrum $S=\{X_n,\pi^m_n\colon m,n\in\omega\}$ of length ω with fully closed projections is scannable.*

Proof. Let $\xi\in\lim_0 S$. The uniqueness of a domination $F_\xi\colon S\to S_\xi$ follows from Lemma 4.4. Existence is established by induction. We already know (from the proof of Lemma 4.4) how to construct the spectrum S_ξ and the domination F_ξ. One only needs to check that this spectrum is Hausdorff. In view of Lemma 2.6 it is sufficient to prove that $d_n\pi^{n+1}_n$ is fully closed at the point $d_n x$, where $x={}^0\pi_{n+1}\xi\in L_{n+1}(S)=X_n$.

This is proved by induction on n: passage from n to $n+1$ is according to Lemma 2.8. Lemma 4.13 is proved.

4.14. Lemma. *Every scannable spectrum $S=\{X_\alpha, \pi_\alpha^{\alpha'}: \alpha, \alpha'<\beta\}$ has fully closed projections. Moreover for every $s \in L_\alpha(S)$ the maps $d^s\pi_\alpha^{\alpha'}$ and so d^s are fully closed.*

Proof. Let $x \in X_\alpha = L_{\alpha+1}(S)$ be an arbitrary point and let ${}^0\pi_\alpha^{\alpha+1}x = s \in L_\alpha(S)$. We have

(1) $\pi_s^{s'} d^{s'} = d^s\pi_\alpha^{\alpha'}$ for all $s' \in ({}^0\pi_\alpha^{\alpha'})^{-1}s$;

(2) $\pi_s^{s'}$ is fully closed for all $s' \in ({}^0\pi_\alpha^{\alpha'})^{-1}s$, since it has at most one non-trivial inverse image, viz. $(\pi_s^{s'})^{-1}d^s x$;

(3) the topology of $X_{\alpha'}$ is openly generated on the set $(\pi_\alpha^{\alpha'})^{-1}x = (\pi_\alpha^{\alpha'})^{-1}(d^s)^{-1}d^s x$ by the family

$$\{X_{\alpha'} \xrightarrow{d^{s'}} X^{s'}: s' \in ({}^0\pi_\alpha^{\alpha'})^{-1}s\}$$

according to the properties (b) and (c) from Definition 4.2.
Hence the map $d^s\pi_\alpha^{\alpha'}$ is fully closed at the point $d^s x$, because of Lemma 2.13. So by Lemma 2.4 the map $\pi_\alpha^{\alpha'}$ is fully closed at the point $x=(d^s)^{-1}d^s x$.

We shall prove that $d^s\pi_\alpha^{\alpha'}$ and d^s are fully closed, by induction on α. If $\alpha=0$, then d^s is the identity and $d^s\pi_\alpha^{\alpha'} = \pi_\alpha^{\alpha'}$. Now we consider the following commutative diagram

$$\begin{tikzcd}
X_\alpha \arrow[d, "d^s"'] & X_{\alpha+1} \arrow[l, "\pi_\alpha^{\alpha+1}"'] \arrow[d, "d^{s'}"] & X_{\alpha'} \arrow[l, "\pi_{\alpha+1}^{\alpha'}"'] \\
X^s & X^{s'} \arrow[l, "\pi_s^{s'}"'] &
\end{tikzcd}$$

where ${}^0\pi_\alpha^{\alpha+1}s' = s$. We write $f=\pi_{\alpha+1}^{\alpha'}$, $g=d^s\pi_\alpha^{\alpha+1}$, $Z=X^s$, $Y=X_{\alpha+1}$, $X=X_{\alpha'}$ and $z=d^s x$, where $x=s' \in L_{\alpha+1}(S)=X_\alpha$. Then applying Lemma 2.8 we obtain that $d^{s'}\pi_{\alpha+1}^{\alpha'}$ is fully closed. This gives an inductive passage from α to $\alpha+1$.

Now we pass to a limit ordinal α. We have already proved that the map $d^s\pi_\alpha^{\alpha'}$ is fully closed at the point $d^s x$, where ${}^0\pi_\alpha^{\alpha+1}x = s$. Let $y \in X^s$, $y \neq d^s x$. By the definition of a scanning and definitions 4.1 and 4.2 the map $\pi_{s''}^s$ is a local homeomorphism at the point y for some $\alpha''<\alpha$ and $s'' = {}^0\pi_{\alpha''}^\alpha s$. Hence $d^s\pi_\alpha^{\alpha'}$ is fully closed at the point y, because $d^{s''}\pi_{\alpha''}^{\alpha'}$ is fully closed at the point $\pi_{s''}^s y$. Lemma 4.14 is proved.

From Lemmas 4.13 and 4.14 we get

4.15. Theorem. *Scannable spectra of length $\leqslant\omega$ are those which have fully closed projections.*

4.16. Corollary. *A map $f: X \to Y$ is fully closed if and only if it is the fibre product of a family $\{Y^y \xrightarrow{\pi^y} Y, y \in Y\}$.*

From the definition and the uniqueness of a domination and from the definition of a roll we obtain two statements.

4.17. Lemma. *If $\{S_\alpha\}$ is an increasing sequence of scannable spectra, then the union $S = \bigcup_\alpha S_\alpha$ is also a scannable spectrum. Moreover $T(S) = \bigcup_\alpha T(S_\alpha)$.*

4.18. Lemma. *If S is a scannable spectrum of a limit length, then its completion $\bar{S}$ is also a scannable spectrum. Moreover $T(\bar{S}) = \overline{T(S)}$.*

Now we shall show that the restriction on the length of the spectrum S in Lemma 4.13 is not accidental.

4.19. Example. A spectrum S with fully closed projections, which is not scannable.

The spectrum S will have length $\omega + 2$. Let $X_n = \{x_0, x_1, \dots, x_{2n+1}\}$, let π_n^{n+1} be a retraction $\pi_0^1 x_2 = x_0$, $\pi_0^1 x_3 = x_1$, and $\pi_n^{n-1} x_{2n+2} = x_{2n-1}$, $\pi_n^{n+1} x_{2n+3} = x_{2n+1}$ for $n \geqslant 1$. Since S has to be continuous, we have $X_\omega = \{x_0, x_1, \dots, x_n, \dots, x_\infty\}$ i.e. X_ω is a convergent sequence. The map π_n^ω is the identity on X_n, sends the set $\{x_{2n+3}, \dots, x_\infty\}$ to the point x_{2n+1} and sends x_{2n+2} to x_{2n-1}. Finally

$$X_{\omega+1} = \{x_0, x_1, \dots, x_n, \dots, x_\infty^-, x_\infty^+\},$$

where $\{x_0, x_1, x_3, \dots, x_{2n-1}, \dots, x_\infty^-\}$ and $\{x_2, x_4, \dots, x_{2n+2}, \dots, x_\infty^+\}$ are convergent sequences. The map $\pi_\omega^{\omega+1}$ sends the points x_∞^- and x_∞^+ to the point x_∞. All maps are fully closed: $\pi_n^{\omega+1}$ has one non-trivial inverse image and the rest $\pi_\omega^{\omega+1}$, π_n^ω are maps into finite spaces.

We assume that there exists a domination $F_\xi: S \to S_\xi = \{Y_\alpha, \rho_\alpha^\beta\}$ with respect to $\xi = x_\infty \in \lim_0 S$. Then $Y_n = \{y_0, y_1, \dots, y_{n+1}\}$, $Y_\omega = \{y_0, y_1, \dots, y_n, \dots, y_\infty\}$. The maps

$$\rho_n^{n+1}: Y_{n+1} \to Y_n, \qquad \rho_n^\omega: Y_\omega \to Y_n$$

are retractions and ρ_n^{n+1} sends y_{n+2} to y_{n+1}. The map $d_0 \in F_\xi$ is a homeomorphism, $d_1 x_0 = y_0$, $d_1 x_1 = y_1$, $d_1 x_2 = y_0$, $d_1 x_3 = y_2$, $d_n | X_{n-1} = d_{n-1}$, $d_n x_{2n} = y_{n-1}$ and $d_n x_{2n+1} = y_{n+1}$. Further $d_\omega | X_n = d_n$ and $d_\omega x_\omega = y_\omega$. The map $f = d_\omega \pi_\omega^{\omega+1}: X_{\omega+1} \to Y_\omega$ coincides with a projection $p: \alpha N \times \{0, 1\} \to \alpha N$, where αN is a convergent sequence. So f is not fully closed at the point y_∞. But $Y_{\omega+1} = (Y_\omega)_f^{y_\omega}$ (see the proof of Lemma 4.4). By Lemma 2.7, $Y_{\omega+1}$ is not Hausdorff, which is a contradiction.

5. The saturated spectral tree and its roll

5.1. Theorem. *Let T be a saturated spectral tree and let $\mathcal{R}(T) = \{S(T), \pi_X : X \in T\}$ be a roll of T. Then $S(T)$ is a scannable spectrum and $T(S(T)) = T$.*

Proof. Induction on the height of T. If $h(T)=1$, then $T=S(T)=T(S(T))$. We assume that the statement is true for all saturated spectral trees of height $<\beta$. Let $h(T)=\beta$. For all $\alpha<\beta$ the tree $T|\alpha$ is saturated and $T|\alpha=T(S(T|\alpha))$.

Let β be a limit ordinal. The sequence $\{S(T|\alpha)\}=\{S(T)|\alpha\}$ is increasing. So $S(T)=\bigcup\{S(T|\alpha):\alpha<\beta\}$ is a scannable spectrum by Lemma 4.17. Moreover $T(S(T))=\bigcup\{T(S(T|\alpha)):\alpha<\beta\}$. But $T(S(T|\alpha))=T|\alpha$. Therefore $\overline{T(S(T))}=T$.

Now let $\beta=(\beta-1)+1$, where $\beta-1$ is a limit ordinal. Then $S(T)=\overline{S(T|\beta-1)}$ by the property (3) of the roll. According to Lemma 4.18 the spectrum $S(T)$ is scannable and

$$T(S(T))=T(\overline{S(T)|\beta-1})=\overline{T(S(T)|\beta-1)}=\overline{T(S(T|\beta-1))}=\overline{T|\beta-1}=T.$$

Finally let $\beta=(\beta-2)+2$. Let $S(T)=\{X_\alpha,\pi_\alpha^{\alpha'}:\alpha,\alpha'<\beta\}$ and let $x\in X_{\beta-2}=\lim_0 S(T)$. By the inductive assumption the spectrum $S(T)|\beta-1=S(T|\beta-1)$ is scannable and $T(S(T)|\beta-1)=T|\beta-1$. Let $\xi={}_0\pi_{\beta-1}x\in\lim_0(S(T)|\beta-1)$. There exist a unique simple spectrum $S_\xi=\{X_\alpha^\xi,{}^\xi\pi_\alpha^{\alpha'}:\alpha,\alpha'<\beta-1\}$ and a unique domination $F_\xi=\{d_\alpha^\xi\}:S(T)|\beta-1\to S_\xi$. Furthermore, S_ξ is a maximal chain of the spectral tree $T|\beta-1$.

We show that $y=d_{\beta-2}^\xi x$ is a distinguished point of $X_{\beta-2}^\xi$. If $\beta=(\beta-3)+3$, then $\xi\in X_{\beta-3}$ and $y\in({}^\xi\pi_{\beta-3}^{\beta-2})^{-1}d_{\beta-3}^\xi\xi$. So y is a distinguished point by property (v) of saturated trees. If $\beta-2$ is a limit ordinal, then by the definition of a domination each point ${}^\xi\pi_\alpha^{\beta-2}y=d_\alpha^\xi\,{}^0\rho_\alpha\xi$, where $\alpha<\beta-2$ and ${}^0\rho_\alpha:\lim_0(S(T)|\beta-1)\to X_\alpha$ is the natural projection, is distinguished. On the other hand by property (iv) there is a successor Y of the space $X=X_{\beta-2}^\xi$. By property (i) we have

$${}^\xi\pi_\alpha^{\beta-2}x_{(X,Y)}={}^\xi\pi_\alpha^{\alpha+1}({}^\xi\pi_{\alpha+1}^{\beta-2}x_{(X,Y)})={}^\xi\pi_\alpha^{\alpha+1}x_{(X_{\alpha+1}^\xi,Y)}=x_{(X_\alpha^\xi,X_{\alpha+1}^\xi)}=\bar{x}_\alpha^\xi={}^\xi\pi_\alpha^{\beta-2}y.$$

Hence $x_{(X,Y)}=y$ in view of the continuity of S_ξ. By property (ii) there is unique space $Y\in T$ such that $y=x_{(X,Y)}$. The space $X_{\beta-1}^x\equiv Y$ defines a maximal chain $S_x=S_\xi\bigcup\{X_{\beta-1}^x,{}^x\pi_{\beta-2}^{\beta-1}\}$ of the tree T.

Let $x'\in X_{\beta-2}$ and $x'\neq x$. If $\xi'={}_0\pi_{\beta-1}x'\neq{}_0\pi_{\beta-1}x=\xi$, then $S_{\xi'}\neq S_\xi$ and so $S_x\neq S_{x'}$. If $\xi'=\xi$, then by property (c) of a domination $d_{\beta-2}^\xi x\neq d_{\beta-2}^\xi x'$ and so $S_x\neq S_{x'}$ (these simple spectra have different distinguished points at level $\beta-2$).

Now let S_ζ be a maximal chain in T. By property (iv) the spectrum S_ζ has length β. Let $y=x_{(X_{\beta-2}^\zeta,X_{\beta-1}^\zeta)}\in X_{\beta-2}^\zeta$. There is a unique point $\xi\in\lim_0(S(T)|\beta-1)$, such that $S(T)|\beta-1$ dominates $S_\zeta|\beta-1$ with respect to ξ. By the definition of a domination $F_\xi=\{d_\alpha^\xi\}:S(T)|\beta-1\to S_\zeta|\beta-1$ we have $|(d_{\beta-2}^\xi)^{-1}y|=1$ and ${}_0\pi_{\beta-1}(d_{\beta-2}^\xi)^{-1}y=\xi$. If we put $x=(d_{\beta-2}^\xi)^{-1}y$, then $S_x=S_\zeta$ according to the construction of S_x.

Thus we have constructed a one-to-one correspondence between $X_{\beta-2}=\lim_0 S(T)$ and the set of all maximal chains S_x of the spectral tree T. It remains only to show that the restriction F_x of the roll $\mathscr{R}(T)$ to S_x is a domination for every $x\in X_{\beta-2}$. Property (a) of 4.2 is satisfied by the definition of a roll. The property (b) is satisfied by the construction of S_x. Finally

$$\{X_{\beta-2}\xleftarrow{\pi_{\beta-2}^{\beta-1}}X_{\beta-1}\xrightarrow{d_{\beta-1}^x}X_{\beta-1}^x:x\in X_{\beta-2}\}$$

is a fan-product of the family

$$\{X_{\beta-2} \xrightarrow{d^x_{\beta-2}} X^x_{\beta-2} \xleftarrow{{}^x\pi^{\beta-1}_{\beta-2}} X^x_{\beta-1} : x \in X_{\beta-2}\}$$

by the construction of $\mathscr{R}(T)$. Hence $(\pi^{\beta-1}_{\beta-2})^{-1}x$ is homeomorphic to

$$\Pi\{({}^{x'}\pi^{\beta-1}_{\beta-2})^{-1} d^{x'}_{\beta-2}x : x' \in X_{\beta-2}\},$$

by Lemma 3.5. But $(d^{x'}_{\beta-2})^{-1} d^{x'}_{\beta-2}x' = x'$. Therefore $d^{x'}_{\beta-2}x$ is not a distinguished point of the spectrum $S_{x'}$, if $x \neq x'$. Hence from the property (iii) we get $|({}^{x'}\pi^{\beta-1}_{\beta-2})^{-1} d^{x'}_{\beta-2}x| = 1$ if $x \neq x'$. Thus $(\pi^{\beta-1}_{\beta-2})^{-1}x$ is homeomorphic to $({}^x\pi^{\beta-1}_{\beta-2})^{-1} d^x_{\beta-2}x$. Theorem 5.1 is proved.

Let $T = \{X, \pi^{X'}_X\}$ and $T' = \{Y, \pi^{Y'}_Y\}$ be spectral trees. The family $M = \{\mu_X : X \in T\}$ is called a *morphism* from T to T' if

(1) μ_X is the map from X onto some $Y \in T'$ with $h(X) = h(Y)$ for every $X \in T$.

(2) if $Y' = \operatorname{codom} \mu_{X'}$, $X < X'$, $Y < Y'$, $h(X) = h(Y)$ then $\operatorname{codom} \mu_X = Y$ and the diagram

(1)
$$\begin{array}{ccc} X & \xleftarrow{\pi^{X'}_X} & X' \\ \Big\downarrow{\scriptstyle \mu_X} & & \Big\downarrow{\scriptstyle \mu_{X'}} \\ Y & \xleftarrow{\pi^{Y'}_Y} & Y' \end{array}$$

is commutative.

If T and T' are spectral trees with distinguished points we demand in addition, that

(3) the image of a distinguished point is a distinguished point and $\mu_X x_{(X,X')} = x_{(Y,Y')}$ for every diagram (1). A morphism $M : T \to T'$ is called an injection, if

(4_i) $X \neq X' \Rightarrow \operatorname{codom} \mu_X \neq \operatorname{codom} \mu_{X'}$.

(5_i) μ_X is a homeomorphism for every $X \in T$. A morphism $M : T \to T'$ is called an epimorphism, if

(4_e) for every $Y \in T'$ there is $X \in T$ such that $Y = \operatorname{codom} \mu_X$.

Let T be a simple spectral tree. A pair (M, T'), where T' is a saturated spectral tree with $h(T') = h(T)$ and $M : T \to T'$ is an injection, is called a saturated extension of T.

A saturated extension $(M_T, \operatorname{Sat} T)$ of T is called a saturation of T, if for each saturated extension (M, T') of T there exists a unique epimorphism $M' : T' \to \operatorname{Sat} T$ such that

$$M_T = M' \circ M$$

5.2. Lemma. *A saturation is unique.*

5.3. Lemma. *If T is a saturated spectral tree, then T is the unique saturated extension of itself. So* Sat $T = T$.

5.4. Lemma. *All connecting projections π^X_{-X} from* (Sat T)$\backslash T$ *are homeomorphisms.*

5.5. Lemma. *If T' is a saturated extension of $T\,|\,\alpha$, then there is a saturated extension T'' of T, such that $T''\,|\,\alpha = T'$.*

5.6. Lemma. $\mathrm{Sat}(T\,|\,\alpha) = \mathrm{Sat}\ T\,|\,\alpha$.

5.7. Lemma. $\mathrm{Sat}\ \bar{T} = \overline{\mathrm{Sat}\ T}$.

5.8. Theorem. *Every simple spectral tree T has a unique saturation* Sat T.

Proof is by induction on $h(T)$. The passage to a limit ordinal β is according to Lemmas 5.2 and 5.5. The passage from a limit ordinal β to $\beta+1$ is according to Lemma 5.7. To pass from $\beta+1$ to $\beta+2$ we add some homeomorphisms.

Slightly changing the proof of Theorem 5.1 we get

5.9. Theorem. *If T is a simple spectral tree, then $S(T)$ is a scannable spectrum and* $T(S(T)) = \mathrm{Sat}\ T$.

Let $T = \{X, \pi^Y_X\}$ be a simple spectral tree. A distinguished point $x_{(X,Y)} \in X \in T$ is called trivial if $|(\pi^Z_X)^{-1} x_{(X,Y)}| = 1$ for $Z \leq Y$ such that $X = {}^-Z$. We denote by $\delta(T)$ the set of all non-trivial distinguished points of T (according to the definition of a spectral tree with distinguished points, the distinguished points $x_{(X,Y)}$ and $x_{(X,Z)}$ are identical if $(X < Y < Z)$. The set $\delta(T)$ is partially ordered in a natural way: for $x \in X$ and $y \in Y$ we have $x \leq y$ if and only if $X \leq Y$ and $\pi^Y_X y = x$.

Let $S = \{X_\alpha, \pi^\alpha_\beta\}$ be a well-ordered spectrum. A point $x \in X_\alpha$ is called trivial if $|(\pi^{\alpha+1}_\alpha)^{-1} x| = 1$. We denote by $\nu(S)$ the set of all non-trivial points of S. The set $\nu(S)$ is partially ordered in the same way as $\delta(T)$.

5.10. Lemma. If T is a simple spectral tree, then there is an isomorphism $e: \nu(S(T)) \to \delta(T)$ such that $e(x) = d^\xi_\alpha(x)$ for $x \in X_\alpha$ for each $\xi \in \lim_0 S(T)$ with $\bar{\pi}_\alpha \xi = x$, where $\bar{\pi}_\alpha: \lim_0 S(T) \to X_\alpha$ is the natural projection (here we assume that $T \subset \mathrm{Sat}\ T$ and $\mathrm{Sat}\ T = T(S(T))$ by Theorem 5.9).

Proof. First we verify that the definition $e(x) = d^\xi_\alpha(x)$ is valid, that is $d^\xi_\alpha(x) = d^\zeta_\alpha(x)$ if $\bar{\pi}_\alpha \xi = \bar{\pi}_\alpha \zeta = x$. We have $\bar{\pi}_\alpha = {}^0\pi_{\alpha+1} = {}_0\pi_{\alpha+2}$ (see the definition of a scanning $T(S)$). Therefore $d^\xi_\alpha = d^\zeta_\alpha$ by Lemma 4.3.

Conditions (b) and (c) from Definition 4.2 imply the inclusion $e(\nu(S(T))) \supset \delta(T)$. Theorem 5.1 implies the inclusion $e(\nu(S(T))) \subset \delta(T)$. Finally the definition of a domination implies that e is an isomorphism. Lemma 5.10 is proved.

6. Fully separable and C-space

We recall that a map $f: X \to Y$ has countable weight (see [10]), if there is an embedding $\iota: X \to Y \times I^\omega$, such that $f = \pi\iota$, where I^ω is the Hilbert cube and $\pi: Y \times I^\omega \to Y$ is the projection.

6.1. Theorem. *A fully closed map $f: X \to Y$ has countable weight if and only if there is a countable set $C \subset Y$ such that*

(1) $|f^{-1}y| = 1$ *for every* $y \in Y \setminus C$,

(2) $w(f^{-1}y) \leq \omega$ *for every* $y \in C$.

Proof. *Sufficiency.* Firstly we consider the case $C = \{y\}$. There is an embedding $g: f^{-1}y \to I^\omega$. Let $h: X \to I^\omega$ be an extension of the map g. Now we define the map $\iota: X \to Y \times I^\omega$ in the following way

$$\iota x = (fx, hx).$$

Clearly $f = \pi\iota$. Furthermore, ι is an embedding, being an injective map from a compact to a Hausdorff space.

Now the general case. By Corollary 4.16 the family

$$\{X \xrightarrow{f^y} Y^y : y \in Y\}$$

is a fibre product of the family

$$\{Y^y \xrightarrow{\pi^y} Y : y \in Y\}.$$

This means that there is an embedding $\kappa: X \to \prod\{Y^y : y \in Y\}$, such that $f^y = p_y \kappa$, where $p_y : \prod\{Y^z : z \in Y\} \to Y^y$ is the projection. Moreover, if

$$p_C : \prod\{Y^y : y \in Y\} \to \prod\{Y^y : y \in C\}$$

is the projection, then $\kappa_C \equiv p_C \kappa$ is an embedding. Indeed, if $x, x' \in X$ and $fx \neq fx'$, then $fx = \pi^y f^y x \neq \pi^y f^y x'$. Hence $f^y x \neq f^y x'$ and so $p_y \kappa x \neq p_y \kappa x'$ for every y. Therefore $p_C \kappa x \neq p_C \kappa x'$. If $x \neq x'$ and $fx = fx' = y \in C$, then $f^y x \neq f^y x'$, because $f^y | f^{-1}y$ is a one-to-one correspondence. We again have $p_C \kappa x \neq p_C \kappa x'$.

Further, let $\lambda_y : Y^y \to Y \times I^\omega_y$ be an embedding, constructed above, such that $\pi^y = \pi_y \lambda_y$, where I^ω_y is a copy of the Hilbert cube and $\pi_y : Y \times I^\omega_y \to Y$ is the projection. We denote by

$$\lambda : \prod\{Y^y : y \in C\} \to \prod\{Y \times I^\omega_y : y \in C\}$$

the product of embeddings λ_y.

Now let $x \in X$. We have

$$\lambda \kappa_C x = (\lambda_y f^y x)_{y \in C} = ((\pi_y \lambda_y f^y x, q_y \lambda_y f^y x))_{y \in C}$$

where $q_y : Y \times I^\omega_y \to I^\omega_y$ is the projection. We put $q_y \lambda_y f^y x = t_y x$. Hence $\lambda \kappa_C x = ((fx, t_y x))_{y \in C}$. We define two maps. Let

$$\Delta : Y \times \prod\{I^\omega_y : y \in C\} \to \prod\{Y \times I^\omega_y : y \in C\}$$

be the following embedding:

$$\Delta(y, (s_{y'})_{y' \in C}) = ((y, s_{y'}))_{y' \in C}.$$

Further, let $\iota : X \to Y \times \prod\{I^\omega_y : y \in C\}$ be defined in the following way:

$$\iota x = (fx, (t_y x)_{y \in C}).$$

We have $\Delta\iota = \lambda\kappa_C$. Hence ι is an embedding. Finally $f = \pi\iota$, where $\pi: Y \times \prod\{I^\omega_y : y \in C\} \to Y$ is the projection. Thus f has countable weight.

Necessity. Let $f: X \to Y$ be a fully closed map of countable weight. Let $C = \{y \in Y : |f^{-1}y| > 1\}$. We assume that $X \subset Y \times I^\omega$ and $f = \pi|X$, where $\pi: Y \times I^\omega \to Y$ is the projection. Let $\mathscr{B} = \{U_i\}$ be a countable base in I^ω. We denote by J the set of all $(i, j) \in \omega \times \omega$ such that $\bar{U}_i \subset U_j$. Let $V_i = Y \times U_i$. For $(i, j) \in J$ let

$$C_{(i,j)} = \{y \in C : f^{-1}y \cap V_i \neq \emptyset \quad \text{and} \quad f^{-1}y \setminus V_j \neq \emptyset\}.$$

We have $C = \bigcup\{C_{(i,j)} : (i, j) \in J\}$. Indeed, let $y \in C$ and let $x_1, x_2 \in f^{-1}y$, $x_1 \neq x_2$. Since $\mathscr{B}$ is a base in I^ω, there is a pair $(i, j) \in J$ such that $x_1 \in \{y\} \times U_i$, $x_2 \notin \{y\} \times U_j$. It follows that $y \in C_{(i,j)}$.

It remains to prove that all sets $C_{(i,j)}$ are finite. We assume that some $C_{(i,j)}$ is infinite. Let y be a limit point of $C_{(i,j)}$. We put $O_1 = X \cap V_j$, $O_2 = X \setminus \bar{V}_i$. By definition of $C_{(i,j)}$ we have $C_{(i,j)} \subset Y \setminus f^{\#}O_1 \cup f^{\#}O_2$. Hence y is a limit point of the set $Y \setminus f^{\#}O_1 \cup f^{\#}O_2$. On the other hand $\{O_1, O_2\}$ covers X, so $\{y\} \cup f^{\#}O_1 \cup f^{\#}O_2$ is open. This contradiction shows that $C_{(i,j)}$ is finite. Theorem 6.1 is proved.

6.2. Remark. The analysis of this proof shows that for a fully closed map $f: X \to Y$ we have the following formula

$$wf = \sum\{w(f^{-1}y) : y \in Y, |f^{-1}y| > 1\}.$$

We shall say that a spectral tree T is a *C-tree* if all connecting maps $\pi^X_{X} \in T$ have countable weight.

6.3. Definition. A space X will be called a *simple C-space*, if X is a limit of a simple C-spectrum.

Immediately from Definition 6.3 we get

6.4. Lemma. *A space X is a simple C-space if and only if $X = \bigcup\{O_\alpha : \alpha < \beta\}$, where*

(1) *O_α is open for each $\alpha < \beta$;*
(2) *$O_\alpha \subset O_{\alpha'}$, if $\alpha < \alpha' < \beta$;*
(3) *$O_\alpha = \bigcup\{O_{\alpha'} : \alpha' < \alpha\}$ for each limit ordinal $\alpha < \beta$;*
(4) *$O_{\alpha+1} \setminus O_\alpha$ has a countable base for its subspace topology for each α with $\alpha + 1 < \beta$.*

6.5. Corollary. *Every closed subspace of a simple C-space is a simple C-space.*

6.6. Definition. A space X will be called a *C-space*, if $X = \lim S(T)$, where T is a simple C-tree.

6.7. Definition. A space X will be called *fully separable* if X is a hereditarily separable C-space.

6.8. Question. It is true, that X is fully separable if and only if every closed subset of X admits an irreducible fully closed map to some metrizable space?

6.9. Lemma. *Every simple C-space X with* $cc(X) \leqslant \omega$ *is fully separable.*

Proof. Let $X = \bigcup\{O_\alpha: \alpha < \beta\}$, where the sequence $\{O_\alpha\}$ satisfies the conditions (1)–(4) from Lemma 6.4, and let $U_\alpha = \operatorname{Int}(O_{\alpha+1} \backslash O_\alpha)$. Since $c(X) \leqslant \omega$, the set $U = \bigcup\{U_\alpha: \alpha + 1 < \beta\}$ has a countable base. Now we shall show that $X \backslash U$ is nowhere dense in X. We assume that $V = \operatorname{Int}(X \backslash U) \neq \emptyset$. Let $\alpha = \min\{\alpha': V \cap O_{\alpha'} \neq \emptyset\}$, In this case α is isolated by condition (3) from Lemma 6.4. Since $V \cap O_{\alpha-1} = \emptyset$, we have $V \cap O_\alpha \subset O_\alpha \backslash O_{\alpha-1}$. This contradicts the condition $V \cap U_{\alpha-1} = \emptyset$. So $X \backslash U$ is nowhere dense in X.

Let Y be the quotient space of X with respect to the decomposition, whose only non-trivial element is $X \backslash U$. The simple quotient map $f: X \to Y$ is irreducible, because $X \backslash U$ is nowhere dense. Finally Y has a countable base, being Alexandroff's compactification of the locally compact space U with countable base.

Thus we have proved that every simple C-space X with $c(X) \leqslant \omega$ admits an irreducible fully closed map onto a space with a countable base. But every closed subset F of a simple C-space with $cc(X) \leqslant \omega$ is a simple C-space with $c(F) \leqslant \omega$. Lemma 6.9. is proved.

6.10. Counterexample ($\neg$SH). There exists a perfectly normal non-separable C-space X.

Construction. Let $T = \{Y, \pi_Y^{Y'}: Y, Y' \in P\}$ be a simple spectral tree such that:

(1) P is a Souslin tree, in which every element has infinitely many followers;

(2) for every distinguished point $x_{(^-Y, Y)}$ the inverse image $(\pi_{\,Y}^{\,^-Y})^{-1} x_{(^-Y, Y)}$ is homeomorphic to the unit interval and has non-empty interior.

(3) the set of distinguished points is dense in $(\pi_{\,Y}^{\,^-Y})^{-1} x_{(^-Y, Y)}$ for every $^-Y \in P$.

Since P is uncountable, the condition (2) implies non-separability of $X = \lim S(T)$. If $S(T) = \{X_\alpha, \pi_\alpha^{\alpha'}: \alpha, \alpha' < \omega_1\}$, then every X_α is metrizable. Every open set $U \subset X$ is the union of F_σ-sets $U_\alpha = \pi_\alpha^{-1} \pi_\alpha^{\#} U$, $\alpha < \omega_1$. If the sequence $\{U_\alpha\}$ is not stable, we can find an uncountable anti-chain in P. Indeed suppose there is an uncountable set $A \subset \omega_1$ such that $U_{\alpha+1} \backslash U_\alpha \neq \emptyset$ for each $\alpha \in A$. It follows that

$$\pi_{\alpha+1}^{-1} \pi_{\alpha+1}^{\#} U \backslash \pi_{\alpha+1}^{-1} (\pi_\alpha^{\alpha+1})^{-1} \pi_\alpha^{\#} U \neq \emptyset \quad \text{or} \quad \pi_{\alpha+1}^{\#} U \backslash (\pi_\alpha^{\alpha+1})^{-1} \pi_\alpha^{\#} U \neq \emptyset.$$

Hence there is a non-trivial point $x_\alpha \notin \pi_\alpha^{\#} U_\alpha$ such that $(\pi_\alpha^{\alpha+1})^{-1} x_\alpha \cap \pi_{\alpha+1}^{\#} U \neq \emptyset$. By Lemma 5.10 and the property (3) of the tree P there exists a non-trivial point

$$y_{\alpha+1} \in (\pi_\alpha^{\alpha+1})^{-1} x_\alpha \cap \pi_{\alpha+1}^{\#} U.$$

If $\alpha, \beta \in A$, $\beta < \alpha$, then $\pi_{\beta+1}^{\alpha+1} y_{\alpha+1} \neq y_{\beta+1}$, because $\pi_\alpha^{\alpha+1} y_{\alpha+1} = x_\alpha \notin \pi_\alpha^{\#} U$ and so

$$\pi_{\beta+1}^{\alpha+1} y_{\alpha+1} = \pi_{\beta+1}^{\alpha} x_\alpha \notin (\pi_{\beta+1}^{\alpha})^{\#} \pi_\alpha^{\#} U = \pi_{\beta+1}^{\#} U.$$

Therefore the set $Y = \{y_{\alpha+1}: \alpha \in A\}$ is an anti-chain. Further we shall assume that A is thin, that is $\alpha \geqslant \beta + 2$ for every pair $\beta < \alpha$ from A. By Lemma 5.10 the set $e(Y)$ is also an anti-chain. For each $z \in e(Y)$ we take $Z \in P$ such that $z = x_{(^-Z, Z)}$. Since all z

have different heights, the set $Q=\{Z\in P: x_{(^-Z,Z)}\in e(Y)\}$ is uncountable. Let us assume that $Z_1<Z_2$ for some $Z_1, Z_2\in Q$. Since $e(Y)$ is an anti-chain $\pi^{^-Z_2}_{^-Z_1}x_{(^-Z_2,Z_2)}\neq x_{(^-Z_1,Z_1)}$. On the other hand by the definition of a spectral tree with distinguished points

$$\pi^{^-Z_2}_{^-Z_1}x_{(^-Z_2,Z_2)}=\pi^{Z_1}_{^-Z_1}\pi^{^-Z_2}_{Z_1}x_{(^-Z_1,Z_2)}.$$

But it follows from the thinness of A that $Z_1\neq {}^-Z_2$ and so

$$\pi^{Z_1}_{^-Z_1}\pi^{^-Z_2}_{Z_1}x_{(^-Z_2,Z_2)}=\pi^{Z_1}_{^-Z_1}x_{(Z_1,^-Z_2)}=x_{(^-Z_1,Z_1)}.$$

This contradiction shows that Q is an anti-chain. All properties of our counter-example are checked.

Now we shall prove the following statement, which will be useful subsequently.

6.11. Lemma (MA+$\neg$CH). *Every almost perfectly normal, compact, separable space X with $\mathrm{cc}(X)\leqslant\omega$ is a sum of a countable set and a perfect set.*

Proof. Let X^α be an α-derivative set of X. We claim that there is a countable ordinal α such that $X^\alpha=X^{\alpha+1}$. We assume that this is false and consider $X^{\omega_1}=\bigcap\{X^\alpha:\alpha<\omega_1\}$. There are two cases.

Case 1. *X^{ω_1} is perfect.* There is countable sequence of neighbourhoods U_i of X^{ω_1} such that $X^{\omega_1}=\bigcap\{U_i: i\in\omega\}$. Since $\mathrm{cc}(X)\leqslant\omega$, every set $X\backslash X^\alpha$ is countable. On the other hand the open sets $X\backslash X^\alpha$ cover the compact set $X\backslash U_i$. Hence $X\backslash U_i$ is countable and so $X\backslash X^{\omega_1}=\bigcup\{X\backslash U_i: i\in\omega\}$ is countable. But $X\backslash X^{\omega_1}$ is the union of the sequence $\{X\backslash X^\alpha:\alpha<\omega_1\}$, which is strictly increasing. Contradiction

Case 2. *X^{ω_1} has an isolated point x.* Let U be a neighbourhood of x such that $\bar U\cap X^{\omega_1}=\{x\}$. Since $|\bar U|\leqslant\omega_1$, $\chi(x,\bar U)\leqslant\omega_1$. On the other hand $\bar U$ is separable. In this case Martin's axiom implies that there is a sequence, coverging to x (see [8]). Therefore $\chi(x,X)=\omega$, because X is almost perfectly normal and so property (3) may be invoked. Consequently x possesses a countable neighbourhood. But this implies that $x\notin X^\alpha$ for some $\alpha<\omega_1$. Contradiction. Lemma 6.11 is proved.

If we assume Jensen's principle $\diamondsuit$ (see [7]), then there is a fully separable almost perfectly normal space X of cardinality $2^{\mathfrak c}$ (see [5]).

6.12. Theorem (MA+$\neg$CH). *Every compact fully separable, almost perfectly normal space X is perfectly normal and so $|X|\leqslant\mathfrak c$.*

Proof. By Proposition 1.3 it is enough to prove that X is first-countable. Let $x\in X$ and let S_x be a simple spectrum, dominated by $S(T)$, where T is a simple C-tree from 6.6, and let $X_x=\lim S_x$. By Theorem 5.9, $S_x\subset\mathrm{Sat}\ T$. So X_x is a simple C-space. Let $\pi_x: X\to X_x$ be the limit of the domination $F_x: S(T)\to S_x$ and $y=\pi_x x$. Since $\pi_x^{-1}y=x$ (by property (b) of a domination), it is enough to prove that y has a countable base of neighbourhoods. But we shall prove more: X_x is first-countable.

We can assume that $X_x = \bigcup\{O_\alpha : \alpha < h(T)\}$, where the sequence $\{O_\alpha\}$ satisfies conditions (1)–(4) from Lemma 6.4. We shall prove by induction that O_α is first-countable. The set O_1 is second- and so first-countable. The passage to a limit ordinal α is by condition (3). Now we make the passage from α to $\alpha + 1$. One needs to check that every point $z \in O_{\alpha+1} \backslash O_\alpha$ has a countable base of neighbourhoods in X_x or, equivalently, $\pi_x^{-1}z$ is a G_δ-set in X. In view of Lemma 6.11 and Lemma 1.4, it is sufficient to show that every point $u \in (\pi_x^{-1}z)\backslash K$, where K is the perfect kernel of $\pi_x^{-1}z$, has a countable base of neighbourhoods. We assume that there is a point $u \in (\pi_x^{-1}z)\backslash K$ of uncountable character relative to X. Since X is almost perfectly normal, we may invoke property (3) to see that u is isolated in $\pi_x^{-1}z$. There are disjoint sets U and V, open in X, such that $\pi_x^{-1}z \subset U \cup V$ and $U \cap \pi_x^{-1}z = \{u\}$. Since π_x is fully closed (by Lemmas 4.14 and 4.18), there is a neighbourhood W of z such that

$$\bar{W} \subset (\{z\} \cup \pi_x^{\#} U \cup \pi_x^{\#} V) \cap O_{\alpha+1}.$$

Let $Y = \bar{W} \backslash \pi_x^{\#} V$. The set $\pi_x^{\#} U$ contains no sequence converging to z, because any such sequence could be lifted to a sequence in U converging to u. Hence $Y\backslash\{z\}$ is countably compact. Since z has countable character in $O_{\alpha+1}\backslash O_\alpha$, it is isolated in $Y \cap (O_{\alpha+1}\backslash O_\alpha)$ so without loss of generality we can assume $Y \cap (O_{\alpha+1}\backslash O_\alpha) = \{z\}$.

Now we show that the complement of any closed non-compact set $F \subset Y\backslash\{z\}$ is σ-compact. Since Y is compact, $\bar{F}^Y = F \cup \{z\}$ for every closed non-compact set $F \subset Y\backslash\{z\}$. We put $\Phi = F \cup \{z\}$, $Z = \pi_x^{-1} Y$ and $\pi = \pi_x | Z$. Note that u is not isolated in $\pi^{-1}\Phi$ since π is fully closed. Suppose that v is an isolated point in $\pi^{-1}\Phi$. Then $v \neq u$ and v is isolated in $\pi^{-1}\pi v$. But $\pi v \in O_\alpha$ and by the induction hypothesis has a countable base of neighbourhoods. Hence $\chi(v, Z) \leqslant \omega$. Now suppose v is not isolated in $\pi^{-1}\Phi$ and that $v \in (\pi^{-1}\Phi)^1\backslash(\pi^{-1}\Phi)^\gamma$ for $1 < \gamma < \omega_1$; then we still have $\chi(v, Z) \leqslant \omega$ by Property (3) of almost perfect normality. Thus by Lemma 6.12 and Lemma 1.4, $\pi^{-1}\Phi$ is a G_δ-set in Z, so $Z\backslash\pi^{-1}\Phi$ is σ-compact. Therefore $(Y\backslash\{z\})\backslash F = \pi(Z\backslash\pi^{-1}\Phi)$ is σ-compact.

Finally Martin's axiom and the negation of the Continuum Hypothesis imply (see Lemma 6.13) that $Y\backslash\{z\}$ is compact, being a countably compact locally compact separable space, whose closed non-compact subsets have σ-compact complements. So the point z is isolated in Y. But $\pi_x^{-1}Y$ contains a neighbourhood $\pi_x^{-1}W \cap (\{u\} \cup \pi_x^{-1}\pi_x^{\#} U)$ of the point u. Hence u is isolated in X, being isolated in $\pi_x^{-1}z$. This contradicts the inequality $\chi(u, X) \geqslant \omega_1$. Theorem 6.12 is proved.

Lemma. 6.13. (MA+$\neg$CH). *Every countably compact, locally compact separable space, whose closed non-compact subsets have σ-compact complements, is compact.*

Proof. Let X be such a space. We assume that X is not compact. Then there exists an open cover $\mathcal{U}$ of X which contains no countable subcover. Therefore all elements of $\mathcal{U}$ are σ-compact. Since $X = \bigcup \mathcal{U}$ and $\mathcal{U}$ has no countable subcover there is a family $\mathcal{U}_1 = \{U_\alpha : \alpha < \omega_1\}$ of members of $\mathcal{U}$ such that for $\alpha < \omega_1$, $\bigcup_{\gamma\leqslant\alpha} U_\gamma \neq \bigcup_{\gamma\leqslant\alpha+1} U_\gamma$. If

$\bigcup_{\gamma<\omega_1} U_\gamma$ were σ-compact we should have $\bigcup_{\gamma<\omega_1} U_\gamma = \bigcup_{\gamma\leqslant\alpha} U_\gamma$ for some $\alpha<\omega_1$, a contradiction. Since $\bigcup \mathscr{U}_1$ is not σ-compact, $X\backslash\bigcup\mathscr{U}_1$ is compact. So $\mathscr{U}$ contains a subcover $\mathscr{V}$ of cardinality ω_1. Then the point ∞ in the Alexandroff compactification $X\cup\{\infty\}$ of X has character $\leqslant\omega_1$, because all elements of $\mathscr{V}$ are σ-compact. On the other hand $X\cup\{\infty\}$ is separable. In this case Martin's axiom and the negation of the Continuum Hypothesis imply that there is a sequence converging to ∞ (see [8]). This contradicts countable compactness of X. Lemma 6.13 is proved.

Remark. Assuming Martin's axiom and the negation of the Continuum Hypothesis, W.A.R. Weiss proved [13] that every countably compact space, whose closed subsets are G_δ, is compact. The following question arises:

Is it true that Martin's axiom and the negation of the Continuum Hypothesis imply the compactness of every countably compact space, whose closed non-compact subsets are G_δ?

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References

[1] V.V. Fedorčuk, Mappings that do not reduce dimension, Dokl. AN SSSR 185 (1969) 54–57.
[2] V.V. Fedorčuk, Strongly closed mappings, Dokl. AN SSSR 187 (1969) 47–49.
[3] V.V. Fedorčuk, A bicompactum of which all infinite closed subsets are n-dimensional, Mat. Sb. 96 (1975) 41–62.
[4] V.V. Fedorčuk, Compatibility of some theorems of general topology with the axioms of set theory, Dokl. AN SSSR 220 (1975) 786–788.
[5] V.V. Fedorčuk, On cardinality of hereditarily separable compact spaces, Dokl. AN SSSR 222 (1975) 302–305.
[6] V.V. Fedorčuk, Fully closed mappings and a compatibility of some theorems of general topology with the axioms of set theory, Mat. Sb. 99 (1976) 3–33.
[7] R.B. Jensen, The fine structure of the constructible hierarchy, Ann. Math. Logic 4 (1972) 229–308.
[8] V.I. Malyhin and B.E. Šapirovskii, Martin's Axiom and properties of topological spaces, Dokl. AN SSSR 213 (1973) 532–535.
[9] A.J. Ostaszewski, On countably compact, perfectly normal spaces, J. London Math. Soc. (to appear).
[10] B.A. Pasynkov, On the dimension and geometry of mappings, Dokl. AN SSSR 221 (1975) 543–546.
[11] B.E. Šapirovskii, On cardinality of hereditarily normal spaces, Dokl. AN SSSR 225 (1975) 767–770.
[12] N.V. Savinov, On fully closed mappings, Vestnik Mosk. Univ., Math. Mech. 4 (1975) 39–45.
[13] W.A.R. Weiss, Countably compact perfectly normal space may or may not be compact, Notices AMS 22N2 1975, Abstract 75T-632. A.-337.
[14] K. Devlin, Variations on ◇, Univ. of Lancaster, England, preprint.